# Efficient and Simple Gibbs State Preparation of the 2D Toric Code via Duality to Classical Ising Chains

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Based on

arXiv:2504.17405

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Summer School in Quantum Matter,
Granada, 5th September 2025

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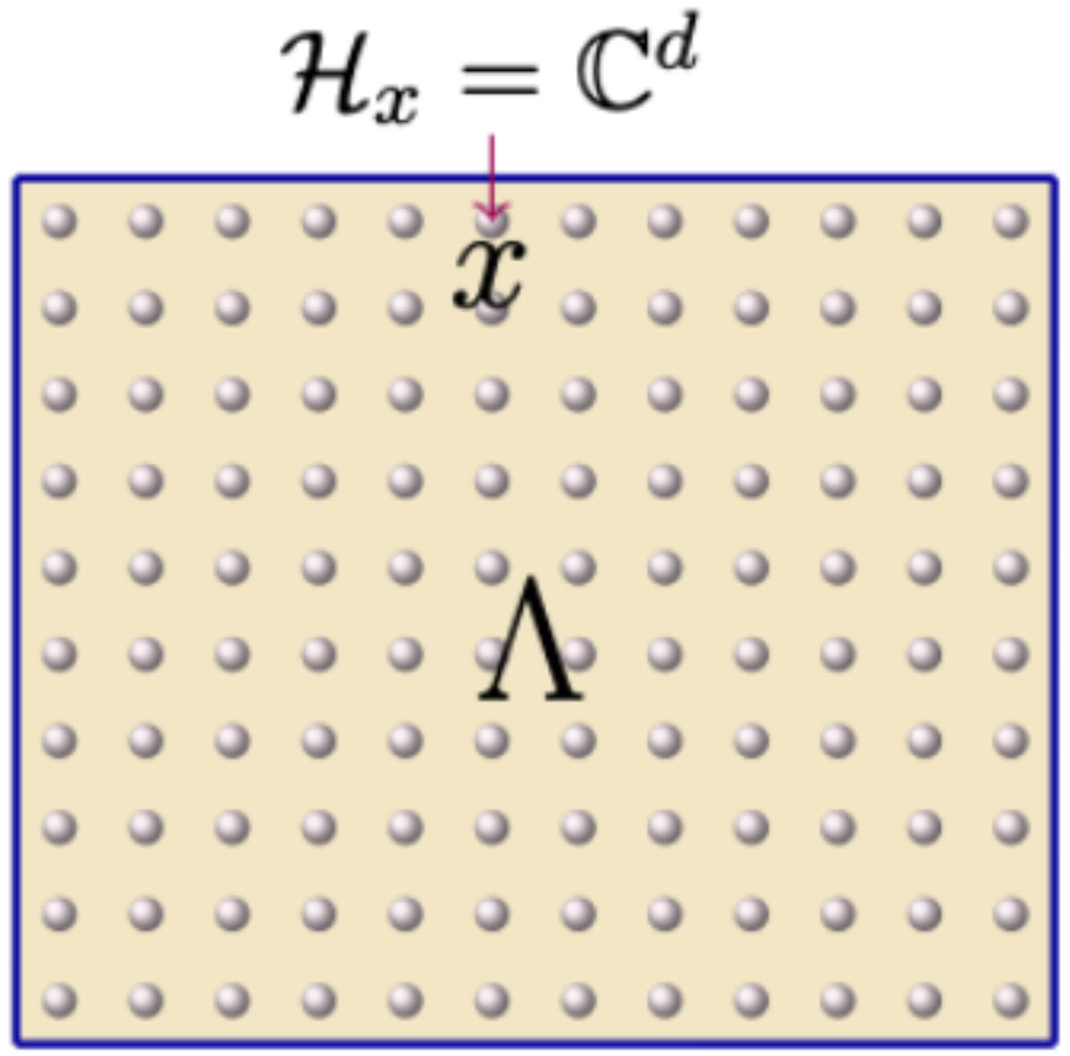
352

# INTRODUCTION TO
# QUANTUM GIBBS SAMPLING

# SETTING: QUANTUM MANY-BODY SYSTEMS

- Spin lattice: $\Lambda \subset\subset \mathbb{Z}^D$
- Hilbert space associated with $\Lambda$ : $\mathscr{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{H}_x \equiv \bigotimes_{x\in\Lambda} \mathbb{C}^d$
- Density matrices: $\mathscr{S}_\Lambda := \mathscr{S}(\mathscr{H}_\Lambda) = \{\rho \in \mathscr{B}(\mathscr{H}_\Lambda) \ : \ \rho \geq 0, \ \mathrm{tr}[\rho] = 1\}$



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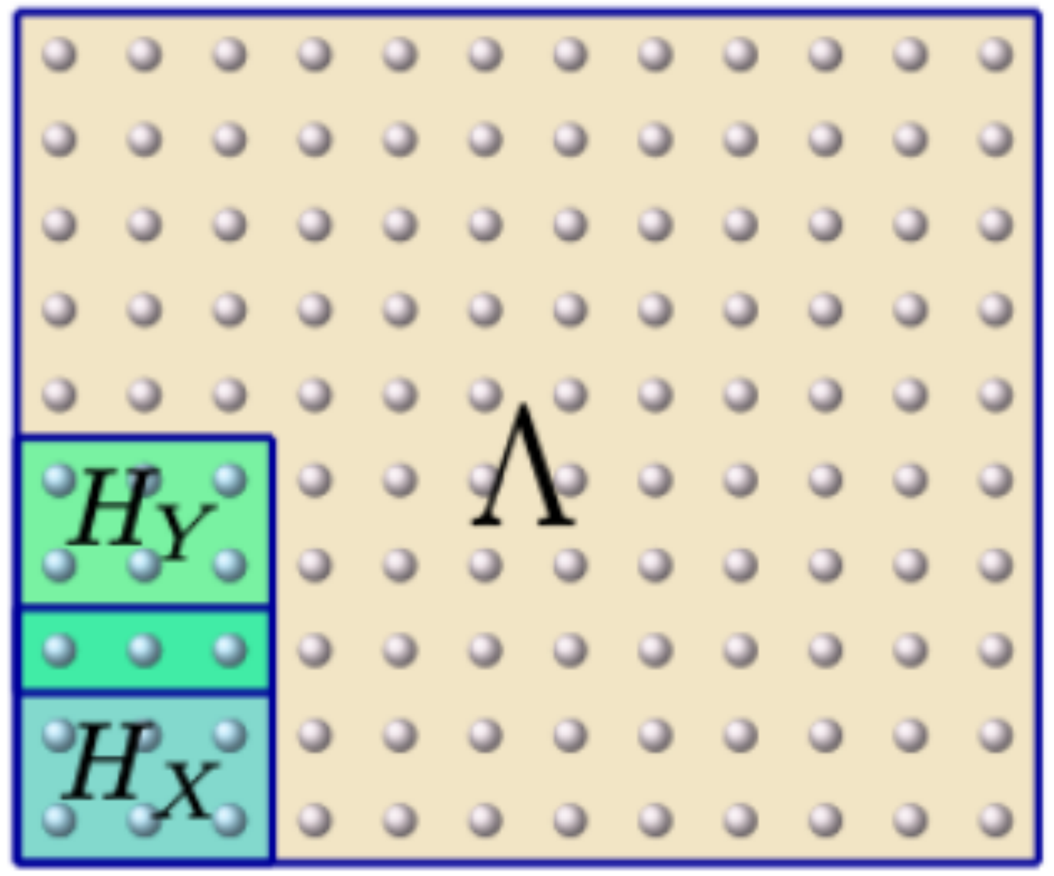
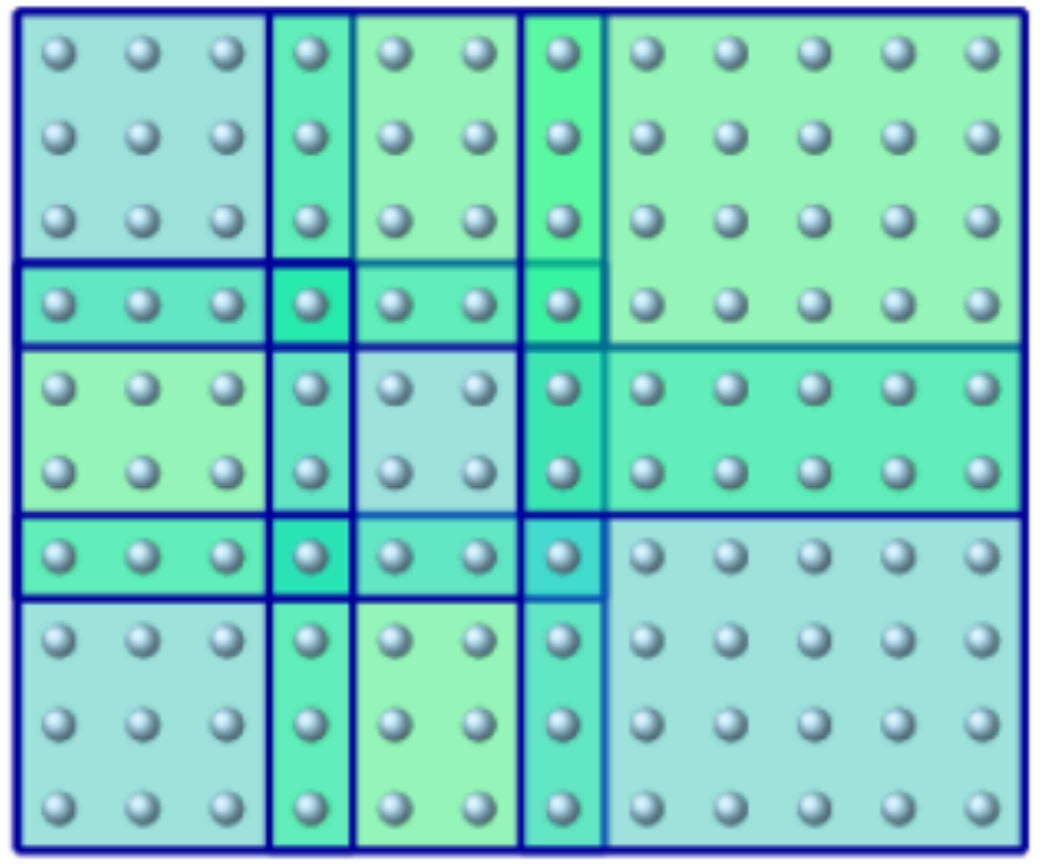
- Hamiltonian: $H_\Lambda = \sum_{X\subset\Lambda} H_X$
- Finite-range (k-local interactions): $\begin{cases} H_X = 0 \text{ for } \mathrm{diam}(X) > k \\ \|H_X\| < J \ \ \forall X \subset \Lambda \end{cases}$





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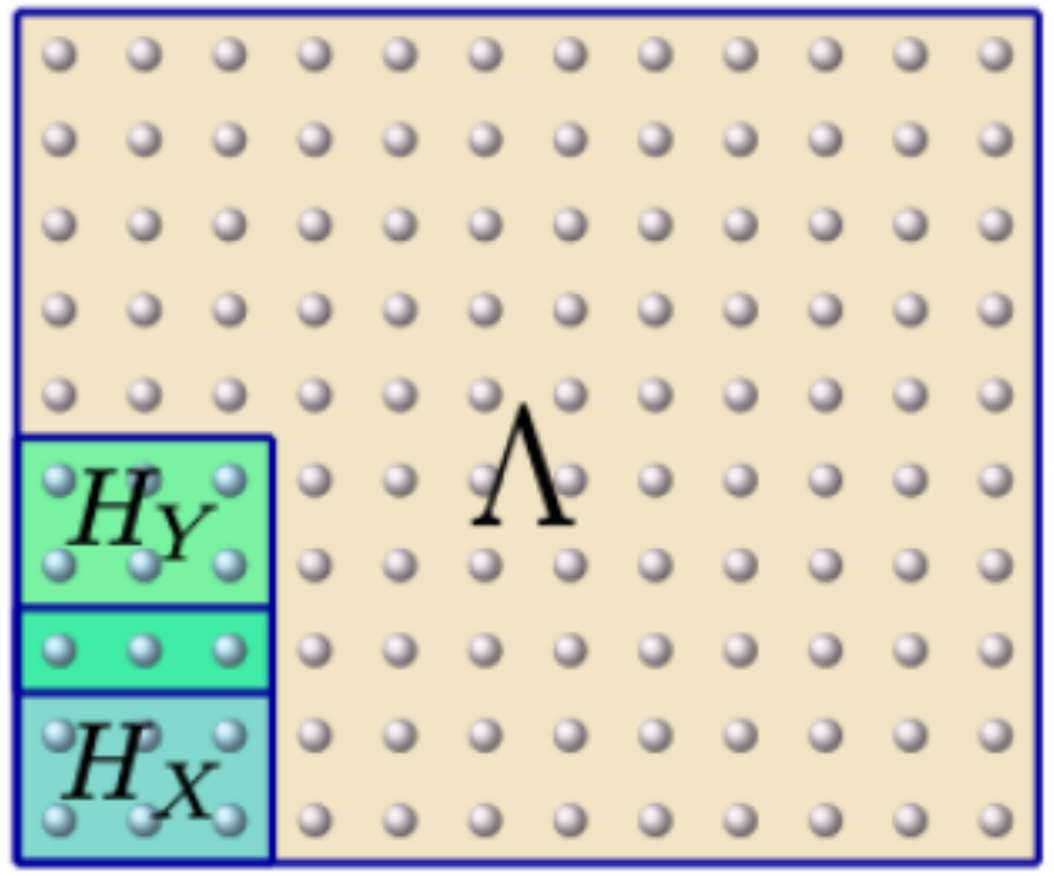

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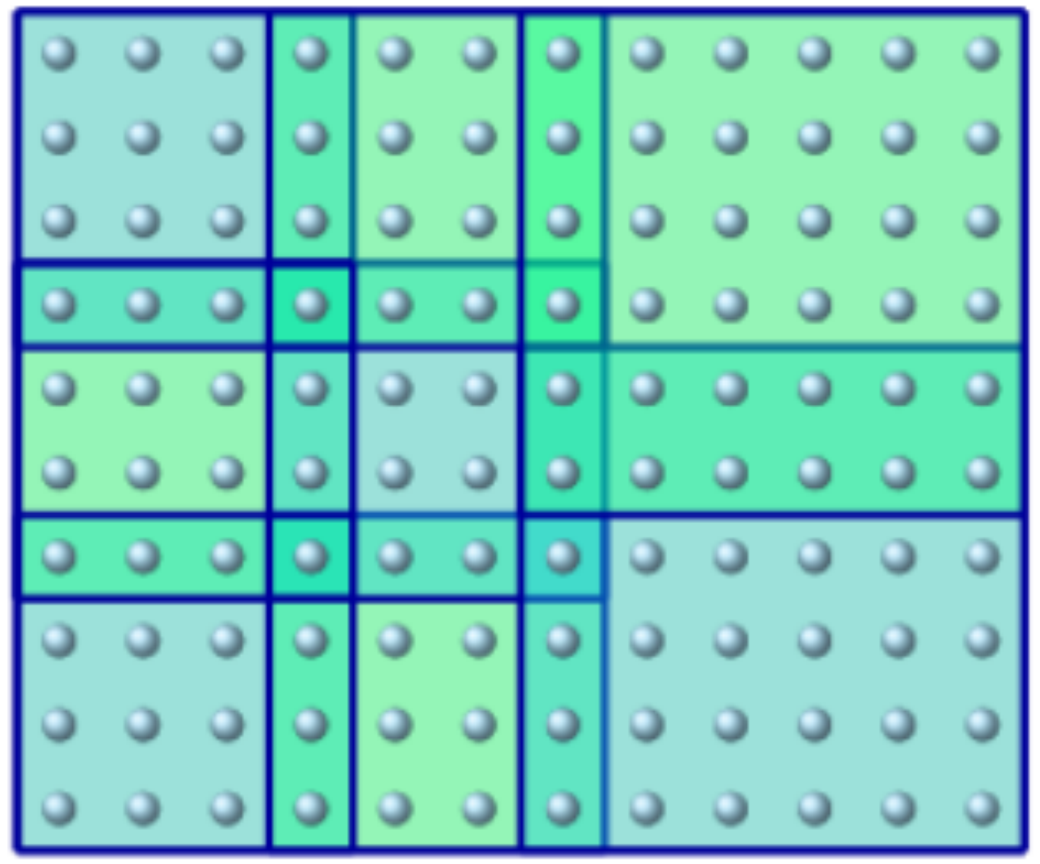

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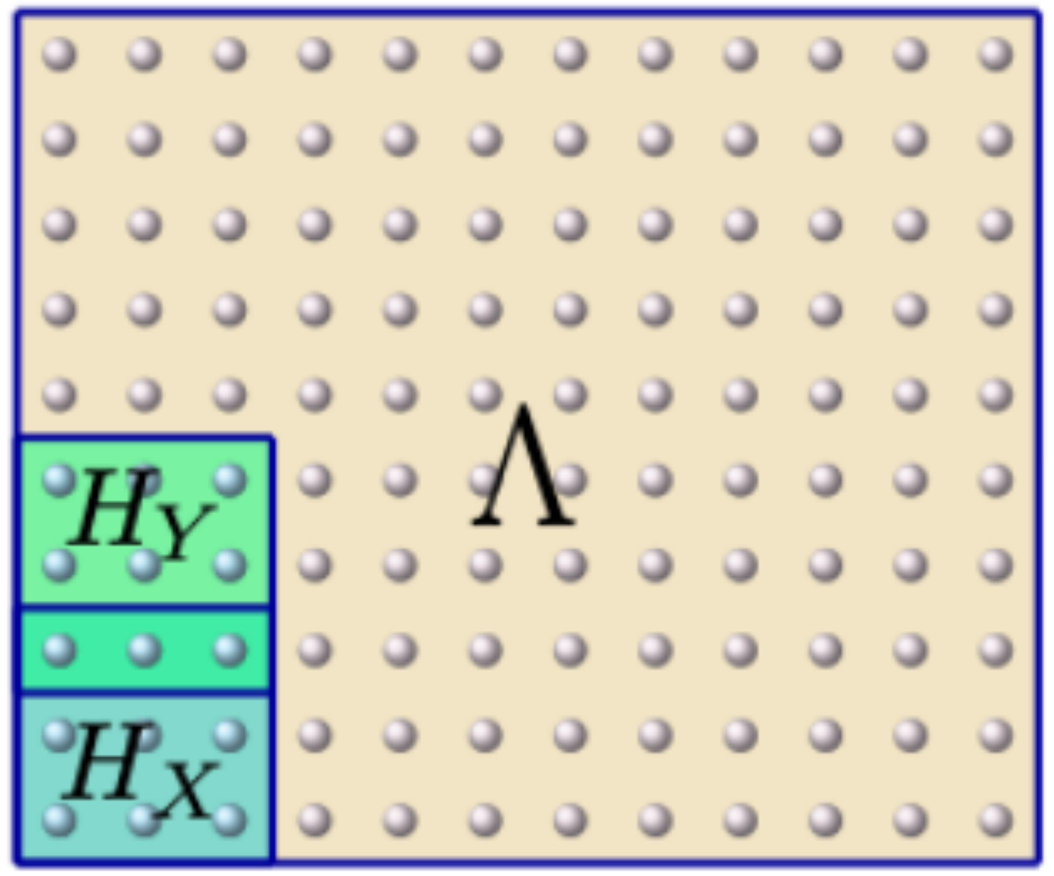

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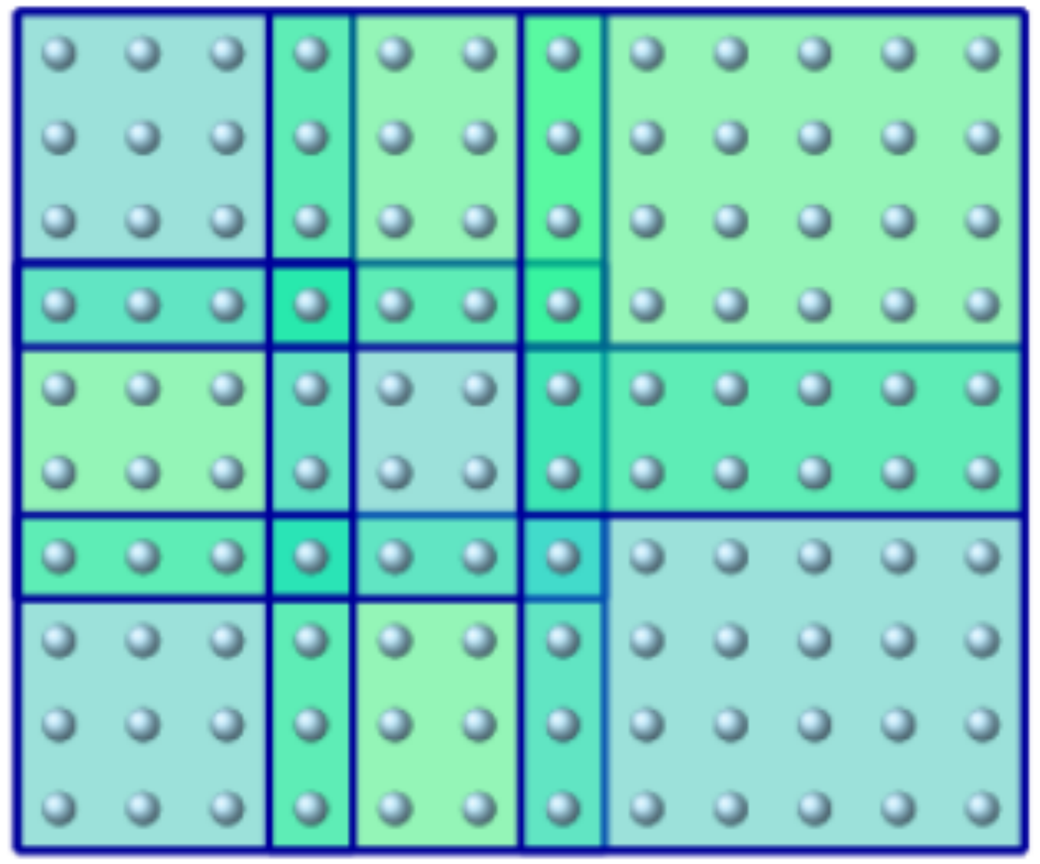

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- Gibbs state (at inverse temperature $\beta > 0$): $\rho^\Lambda := \dfrac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$

# GIBBS SAMPLING / PREPARATION OF GIBBS STATES

$$H_\Lambda = \sum_{X \subset \Lambda} H_X$$

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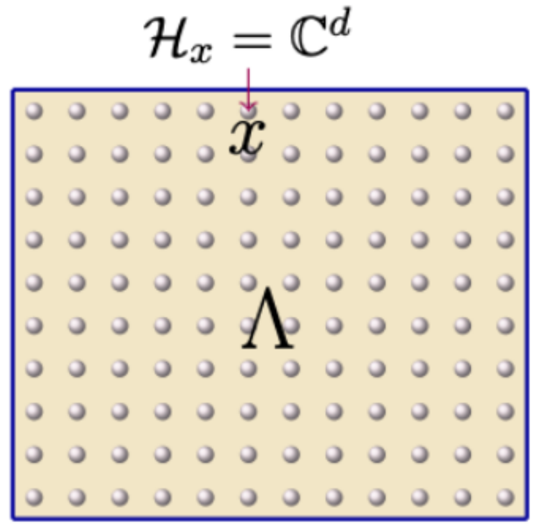

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$$(\, \|\widetilde{\rho} - \rho\| \leq \varepsilon \,)$$

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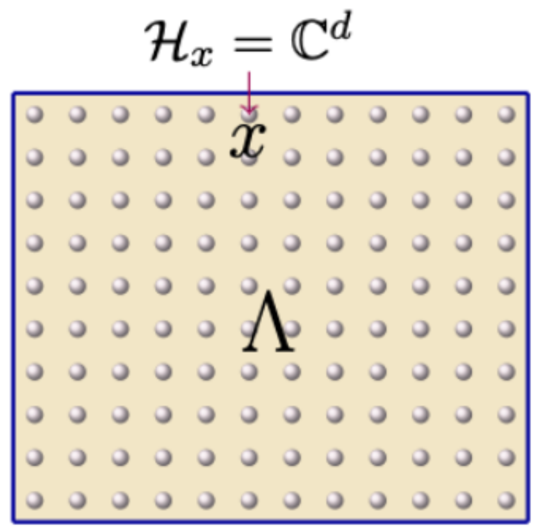

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# Gibbs sampling / Preparation of Gibbs states

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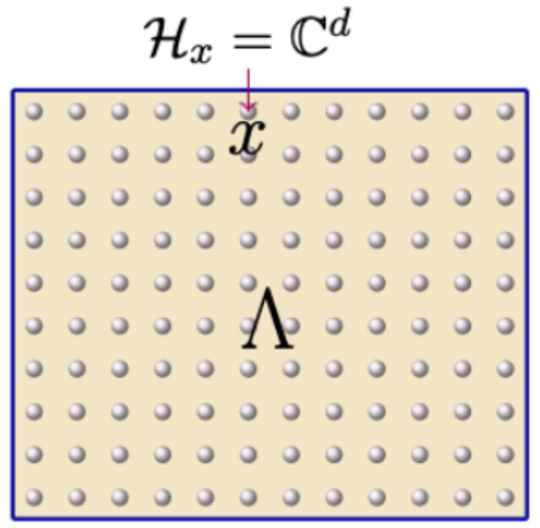

How do we do **Gibbs sampling?**

- A typical way is via **dissipation.**

# QUANTUM DISSIPATIVE EVOLUTIONS

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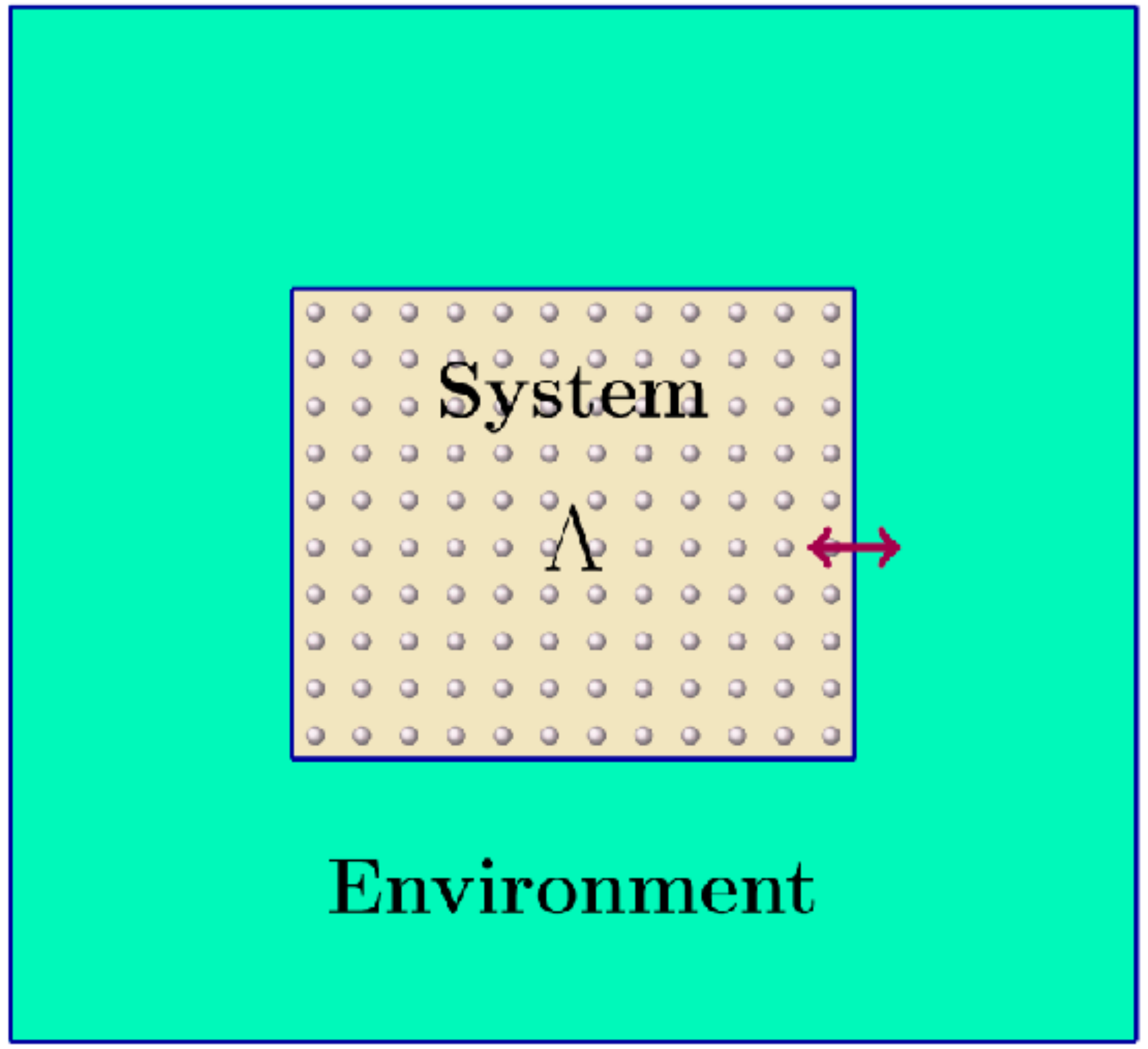

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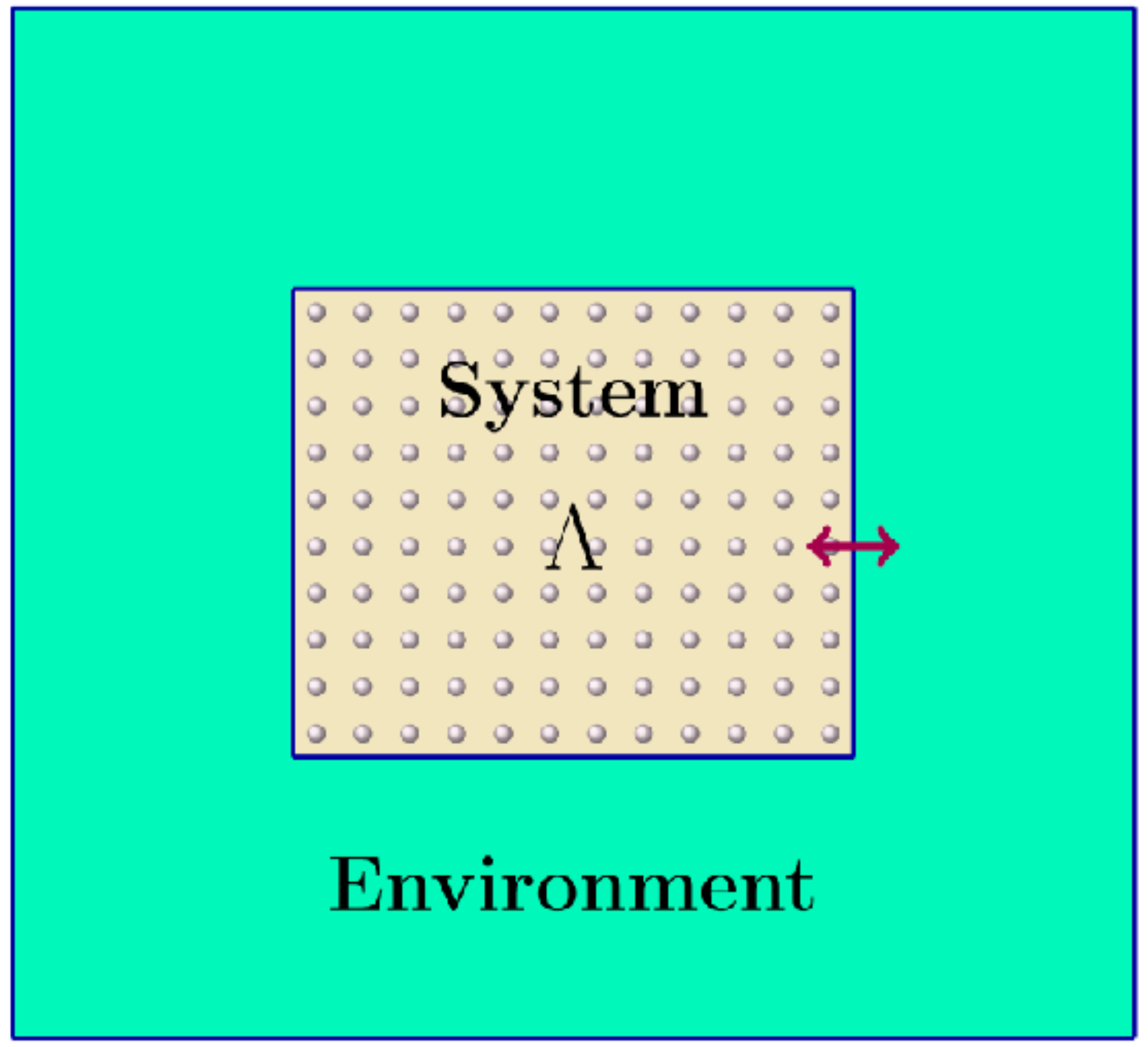

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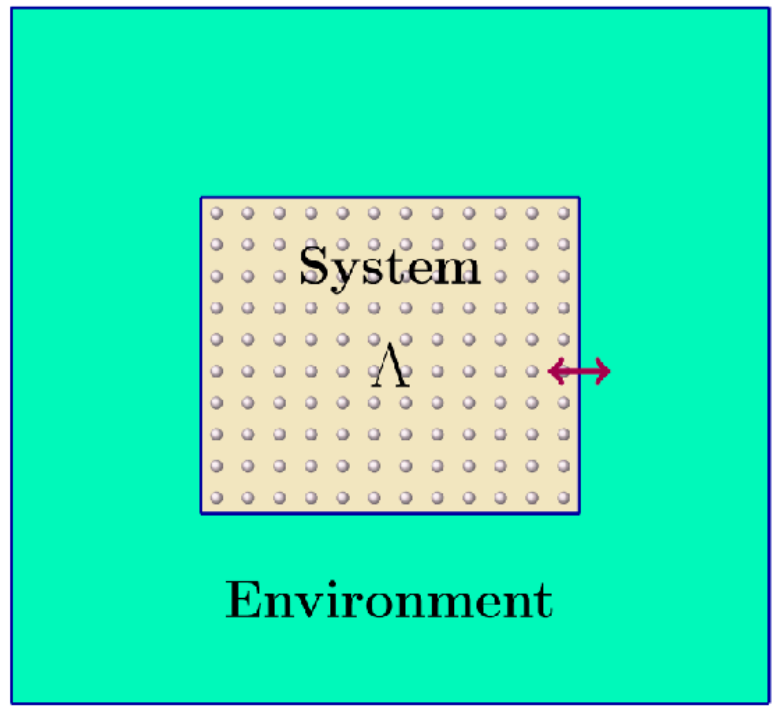

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- Lindbladian: $\mathcal{L}$ describes the dynamics of the system and $\mathcal{L}(\rho) = 0$
- Given $\sigma \in \mathcal{S}(\mathcal{H}_\Lambda)$

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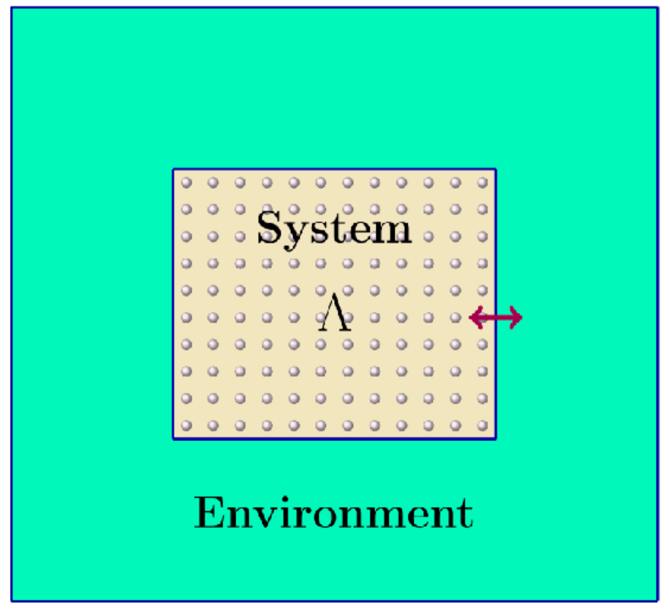

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- Given $\sigma \in \mathcal{S}(\mathcal{H}_\Lambda)$ $\quad \boxed{\mathrm{e}^{t\mathcal{L}}(\sigma) \xrightarrow{t\to\infty} \rho}$

- **Dissipative quantum state engineering:** Robust way of engineering relevant quantum states and algorithms

[Verstraete, Wolf, Cirac, NatPhys '09] [Kraus, Büchler, Diehl, Kantian, Micheli, Zoller, PRA '08]

# Efficient Gibbs Sampling with Dissipation

- Given $\sigma \in \mathcal{S}(\mathcal{H}_\Lambda)$

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# EFFICIENT GIBBS SAMPLING WITH DISSIPATION

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## Efficient preparation of Gibbs states

When do we have $\|\, \mathrm{e}^{t\mathcal{L}}(\sigma) - \rho\|_1 \leq \varepsilon$ ?

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## Ingredients

1. Efficient implementation of the Lindbladian

2. Rapid/fast mixing of the evolution

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1. **Commuting case:** Efficient implementation of Davies generator

   [Rall, Wang, Wocjan, Quantum'23] [Li, Wang ICALP'23]

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# EFFICIENT IMPLEMENTATION OF THE LINDBLADIAN

Number of qubits: $|\Lambda|$

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Circuit complexity: $\mathcal{O}(|\Lambda|^2 \mathrm{polylog}|\Lambda|)$ Circuit depth: $\mathcal{O}(|\Lambda| \mathrm{polylog}|\Lambda|)$

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# RAPID/FAST MIXING OF THE EVOLUTION

$$e^{t\mathcal{L}}(\sigma) \xrightarrow{t\to\infty} \rho$$

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   - 1D, TI, any positive temperature, rapid mixing
     [Bardet, AC, Gao, Lucia, Pérez-García, Rouzé, CMP'23 and PRL'23]
   - High D, 2-local, under decay of correlations + gap, rapid mixing
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Mixing time: $\mathcal{O}(\text{polylog}\,|\Lambda|)$ for rapid mixing, $\mathcal{O}(\sqrt{|\Lambda|}\log|\Lambda|)$ for fast mixing.

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Here we explore another simpler approach for specific models

# QUANTUM GIBBS SAMPLING
# VIA DUALITY

# DUALITY

Consider $H_1$ and $H_2$ two Hamiltonians.

We say that they are **poly-depth dual** if there exists a unitary $U$ that can be implemented by a circuit (of 2-local gates) of polynomial depth such that

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Therefore, if $\rho_1$ can be **efficiently sampled**, $\rho_2$ as well.

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Assume that $\rho_1$ can be efficiently sampled with $\mathscr{C}$ .

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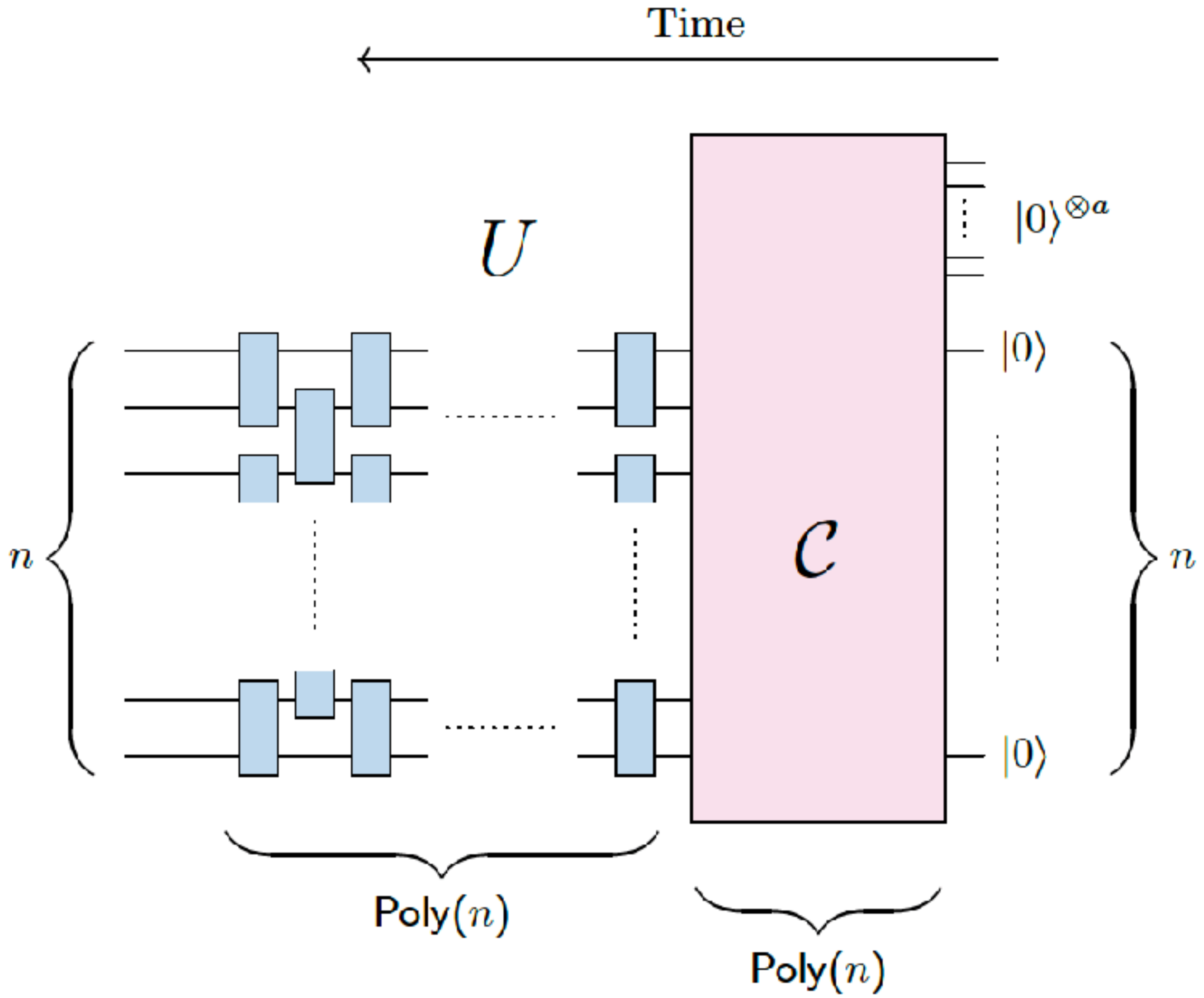

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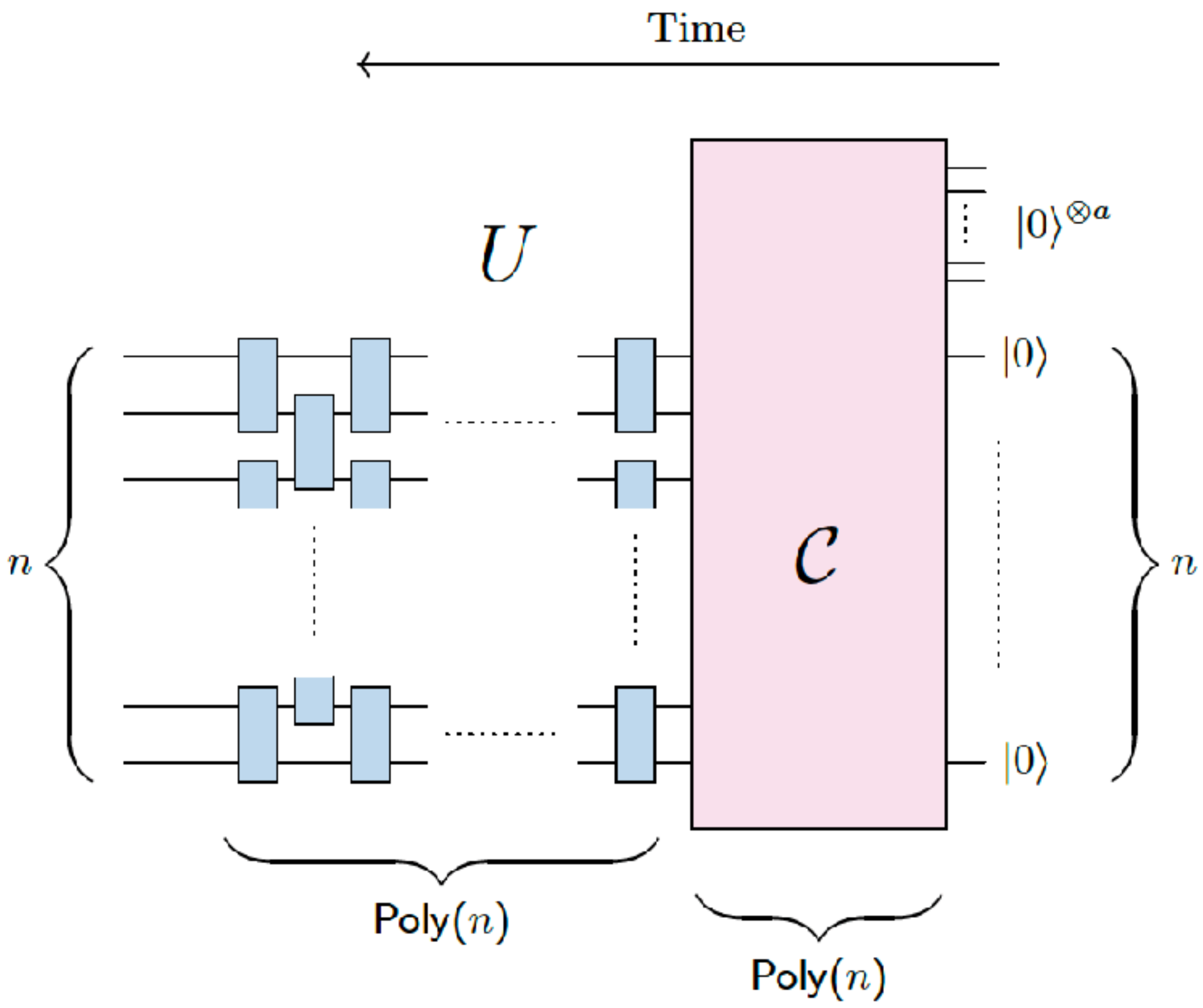


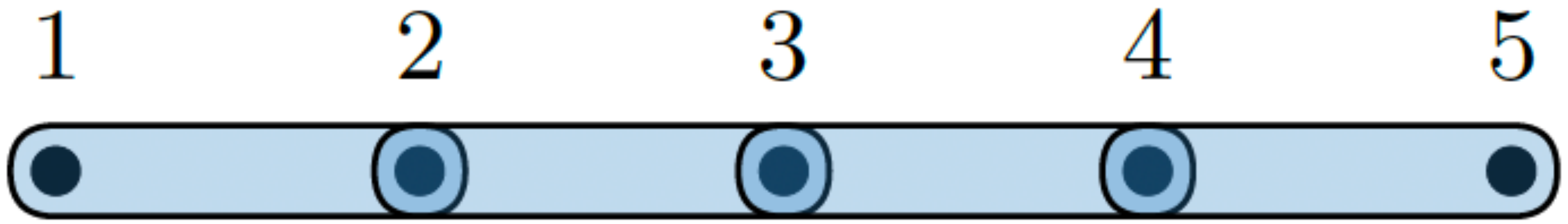
Ingredients. For a relevant Hamiltonian $H_2$:

1. Find a poly-depth circuit mapping it to $H_1$
2. Find an efficient sampler for $\rho_1$

# EXAMPLE: 1D ISING CHAIN

## CLASSICAL 1D ISING CHAIN (OF LENGTH L)

$$H = -\sum_{i=1}^{L-1} J_i \sigma_z^i \sigma_z^{i+1}$$



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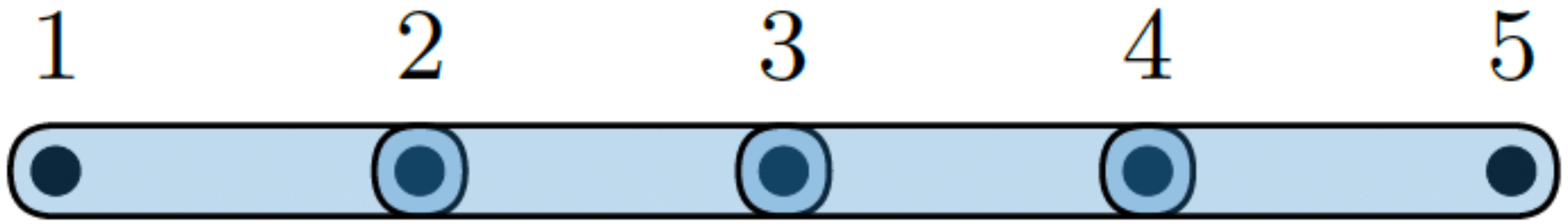

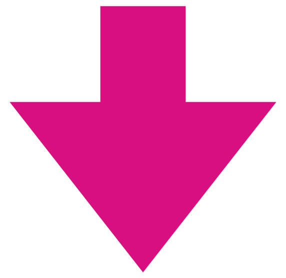

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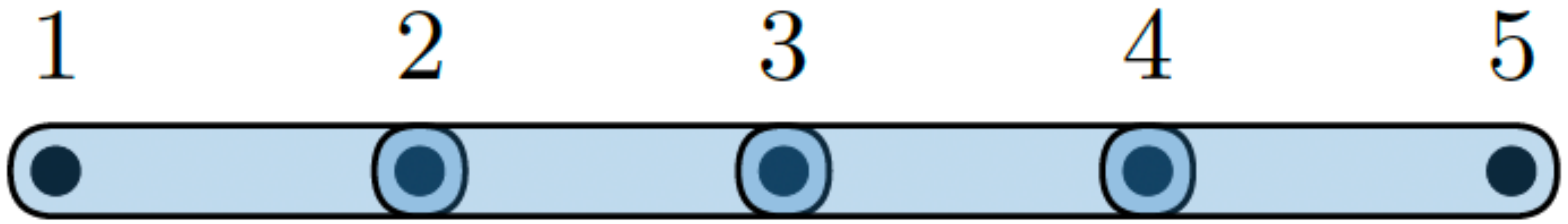

$$U := CX(1,2)\,CX(2,3)\cdots CX(L-1,L)$$

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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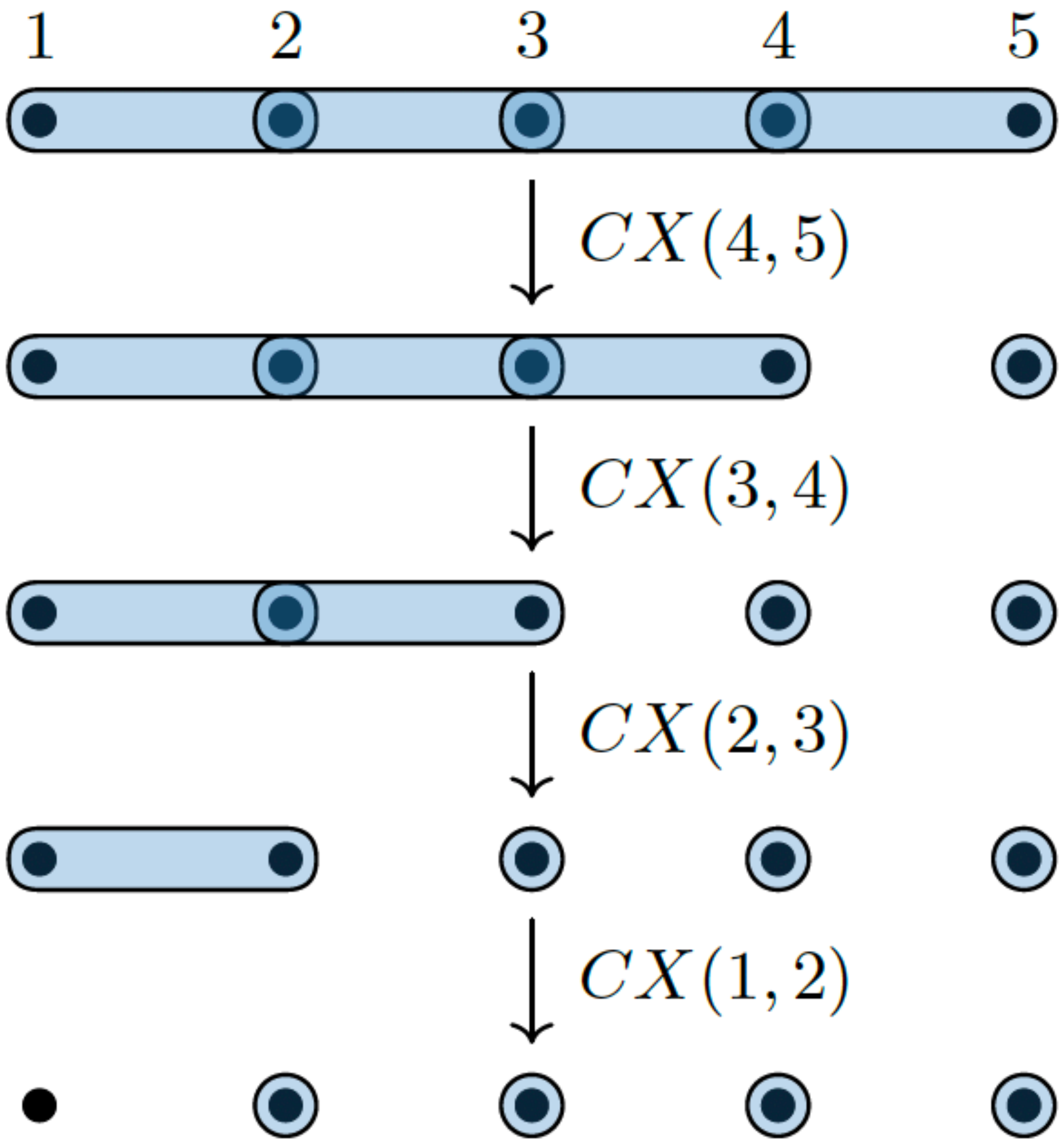

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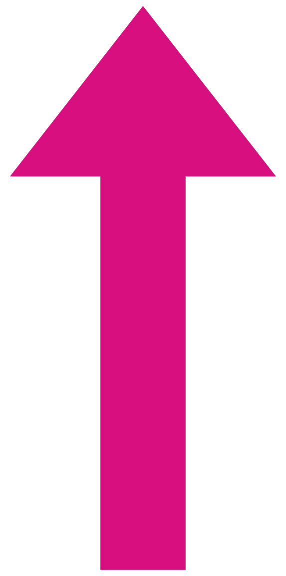

$\mathcal{O}(L)$ depth

## NON-INTERACTING HAMILTONIAN
(OF LENGTH L)

$$UHU^\dagger = -\sum_{i=2}^{L} J_{i-1} \sigma_z^i$$

$\dfrac{e^{-\beta UHU^\dagger}}{\mathrm{Tr}[e^{-\beta UHU^\dagger}]}$ can be sampled in $\mathcal{O}(1)$.

# EXAMPLE: 1D ISING CHAIN

## CLASSICAL 1D ISING CHAIN

## (OF LENGTH L)

$$H = -\sum_{i=1}^{L-1} J_i \sigma_z^i \sigma_z^{i+1}$$

$\frac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$ can be sampled in $\mathcal{O}(L)$ .

$\mathcal{O}(L)$ depth

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## NON-INTERACTING HAMILTONIAN

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# DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

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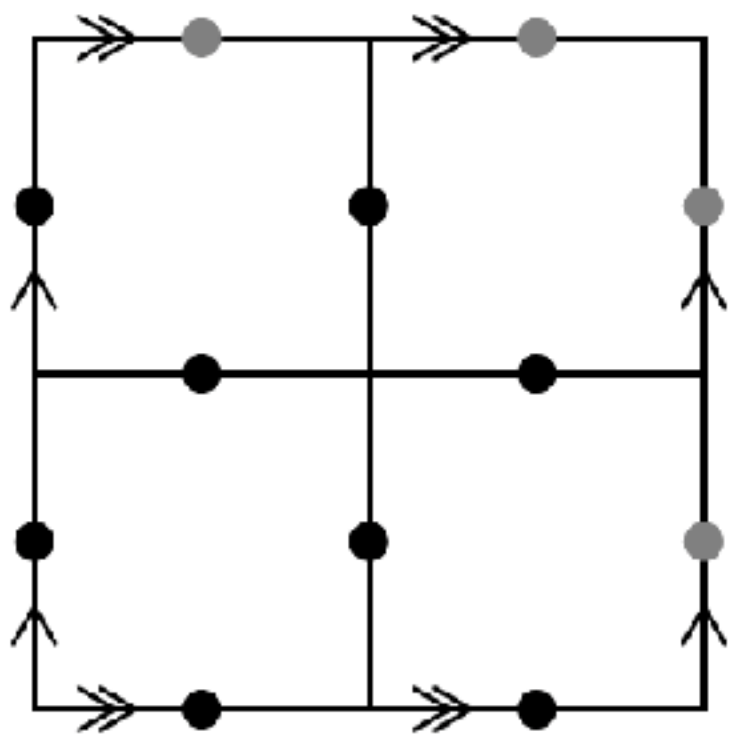
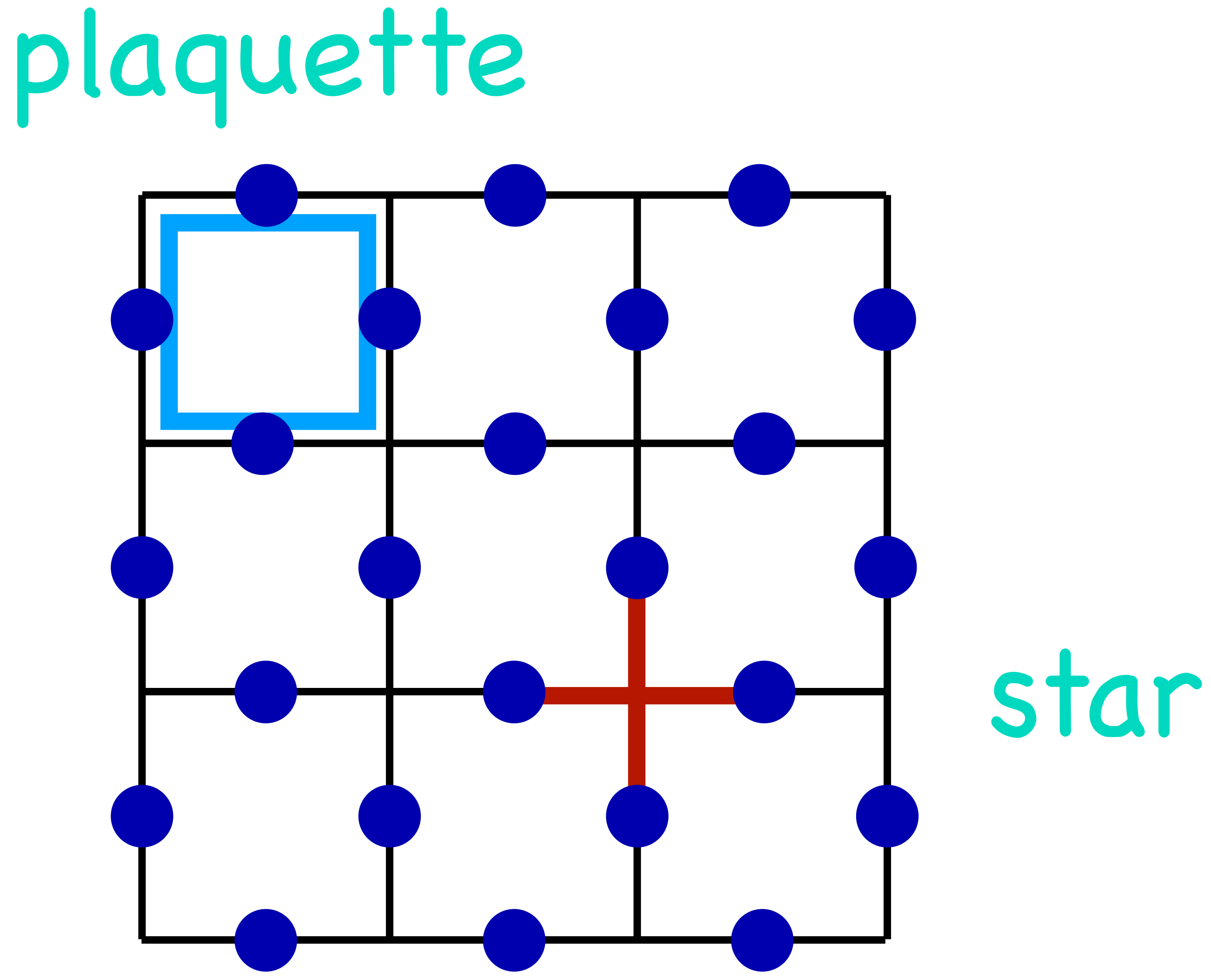
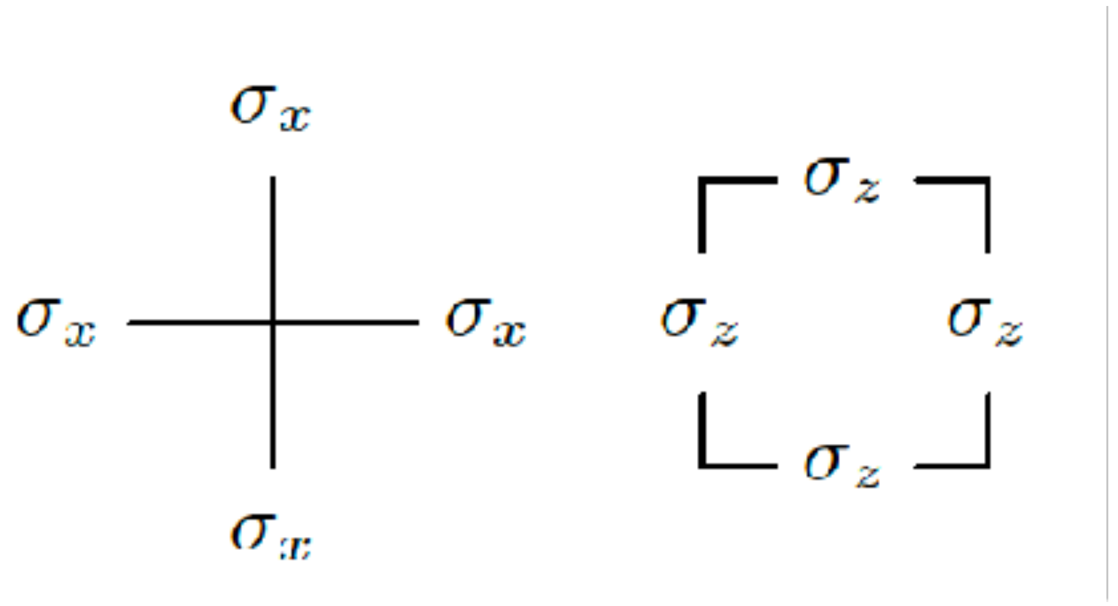
## 2D TORIC CODE

Geometry

Interactions







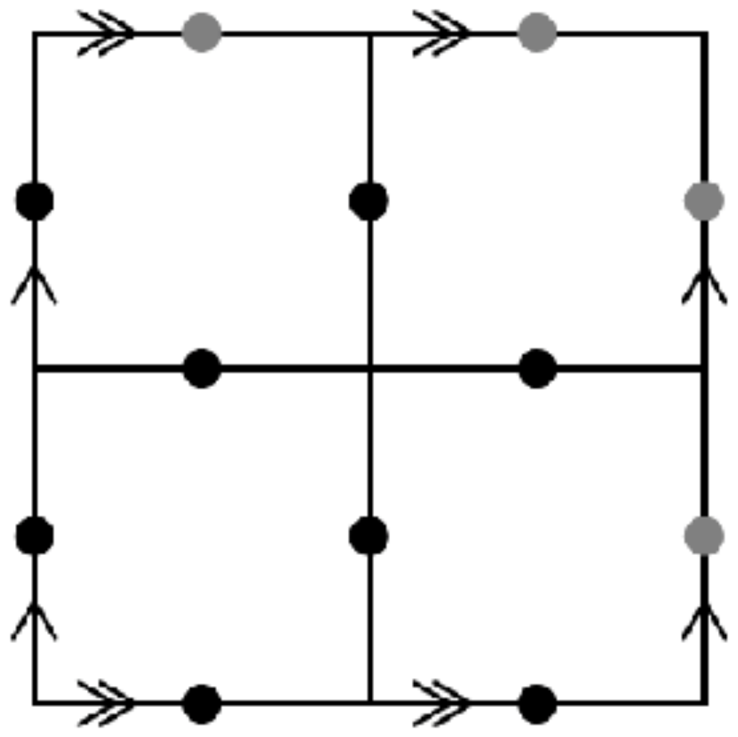
Hamiltonian

$$H_{TC} = -\sum_{v\in V_L} J_v A_v - \sum_{p\subset \mathcal{E}_L} J_p B_p \qquad A_v := \bigotimes_{i\in\partial v} \sigma_x^i, \quad B_p := \bigotimes_{i\in p} \sigma_z^i.$$

# DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

## 2D TORIC CODE

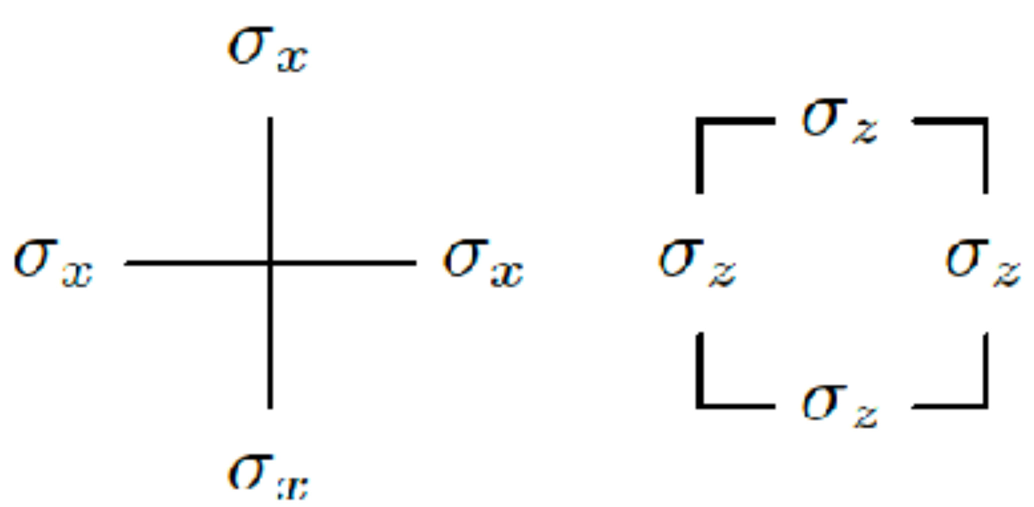
### Geometry

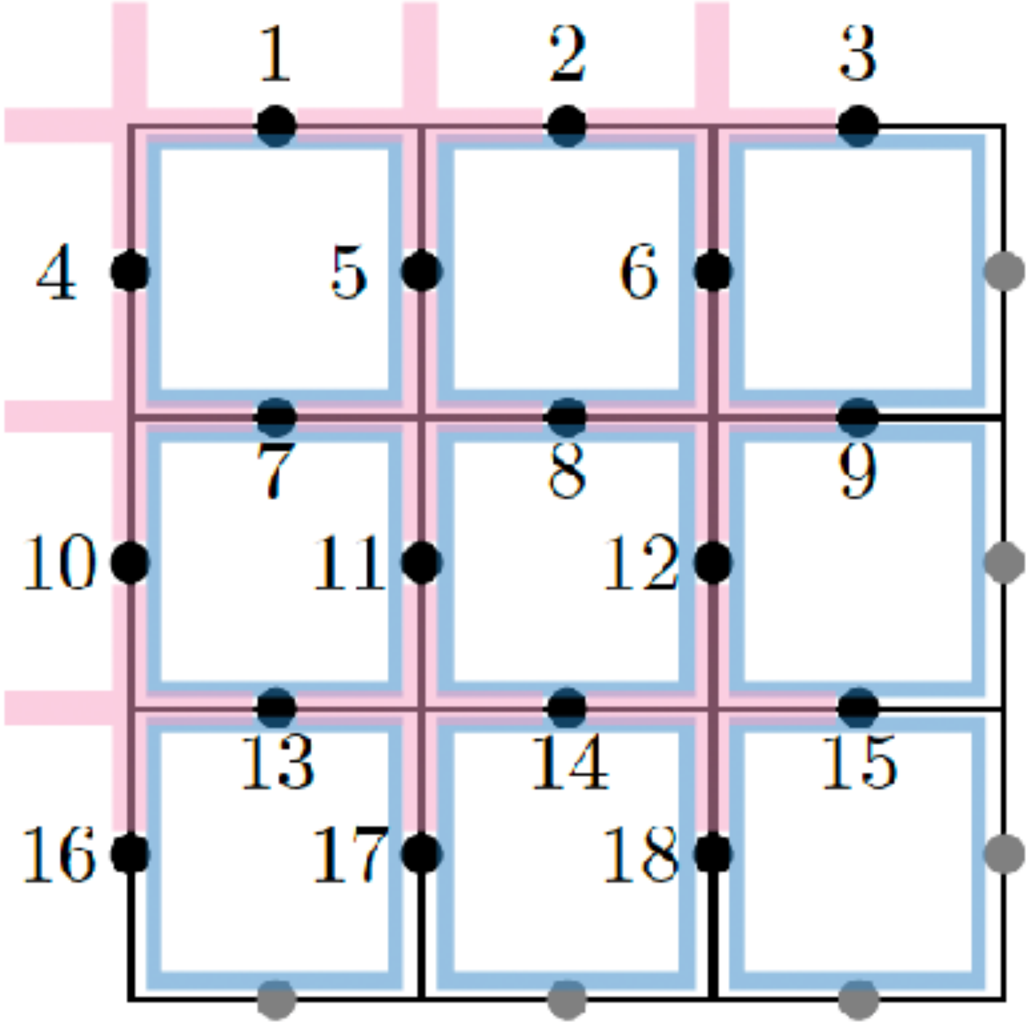


### Interactions

star

plaquette





(for 3x3)

### Hamiltonian

$$H_{TC} = -\sum_{v \in V_L} J_v A_v - \sum_{p \subset \mathcal{E}_L} J_p B_p$$

$$A_v := \bigotimes_{i \in \partial v} \sigma_x^i, \quad B_p := \bigotimes_{i \in p} \sigma_z^i.$$

# DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

MAIN RESULT For the **2D Toric Code** in an $L \times L$ lattice, there exists a **quantum circuit** $C$ composed of $\mathcal{O}(L^3)$ $CX$ gates and $\mathcal{O}(L^2)$ Hadamard gates such that

$$C\Big(\sum_{v \in V_L} J_v A_v\Big)C^\dagger \text{ and } C\Big(\sum_{p \subset \mathcal{E}_L} J_p B_p\Big)C^\dagger$$

correspond to **2 disjoint 1D Ising chains.**

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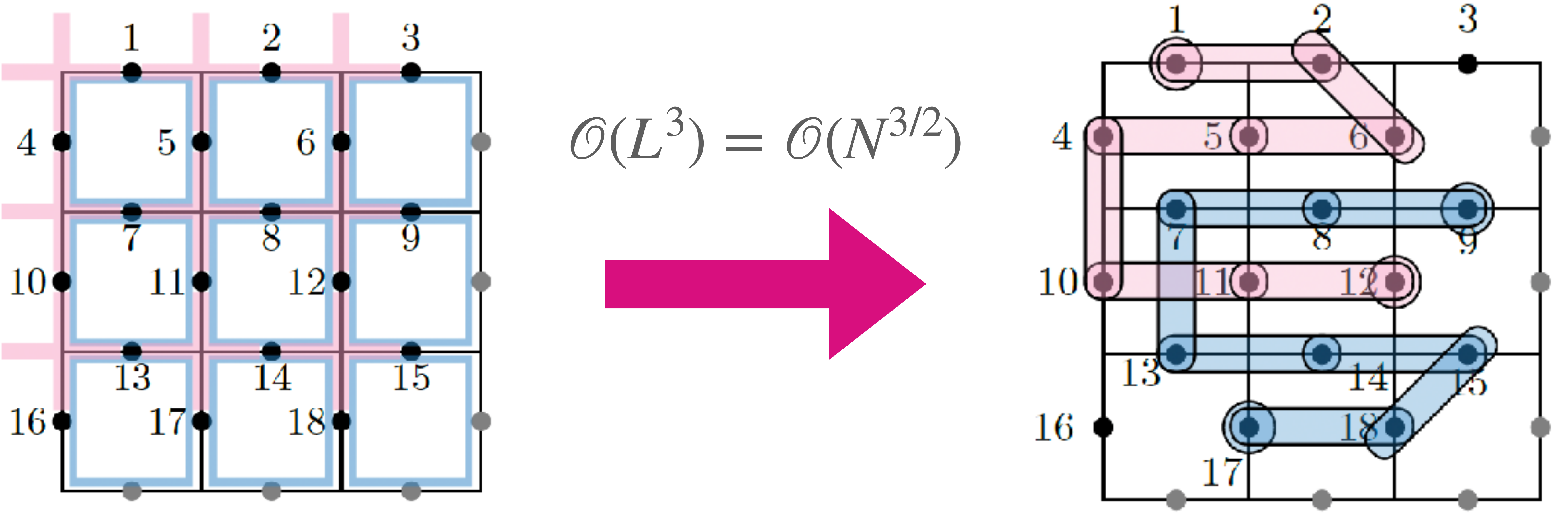

$$\mathcal{O}(L^3) = \mathcal{O}(N^{3/2})$$

# DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

## MAIN RESULT

For the **2D Toric Code** in an $L \times L$ lattice, there exists a **quantum circuit** $C$ of complexity $\mathcal{O}(L^3)$ such that

$$C\Big(\sum_{v\in V_L} J_v A_v\Big)C^\dagger \text{ and } C\Big(\sum_{p\subset \mathcal{E}_L} J_p B_p\Big)C^\dagger$$

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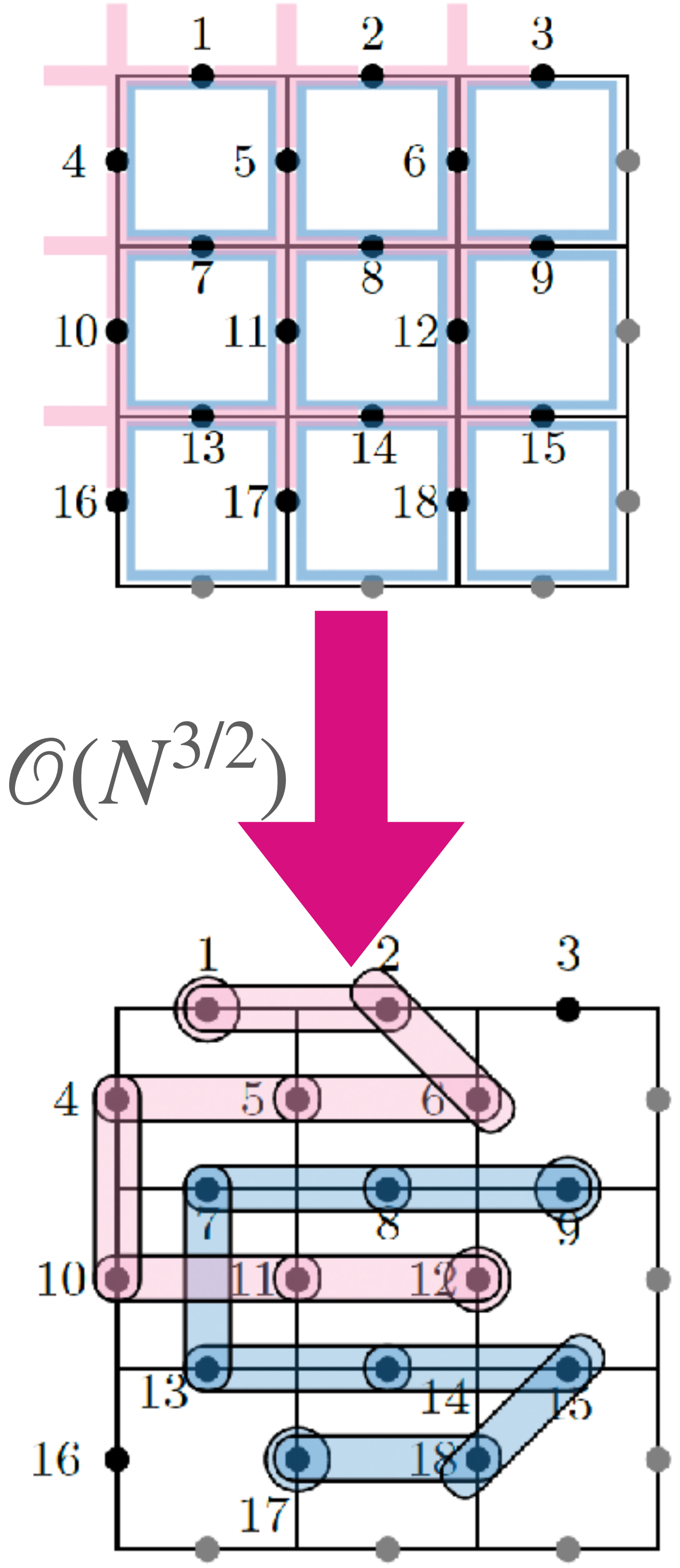
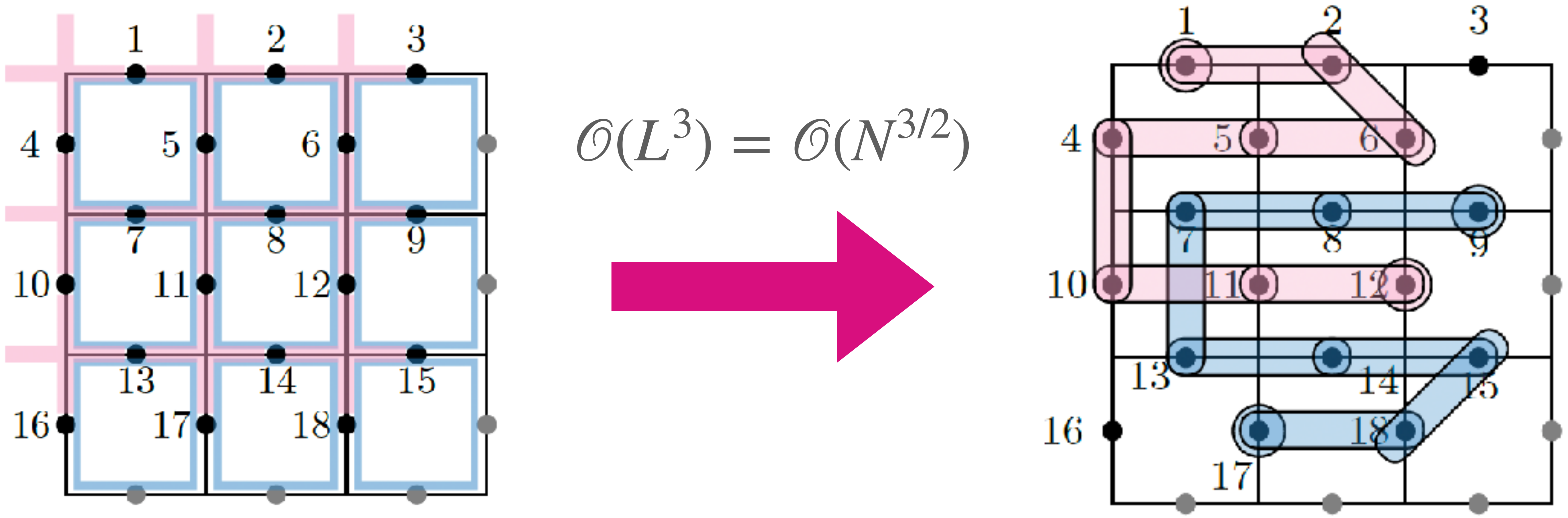


## CONSEQUENCE

The **ground** and **Gibbs state** of the 2D Toric Code can be prepared with a **gate complexity** of $\mathcal{O}(L^3)$ for any $0 \leq \beta \leq \infty$ .

# DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

## STEPS OF THE PROOF



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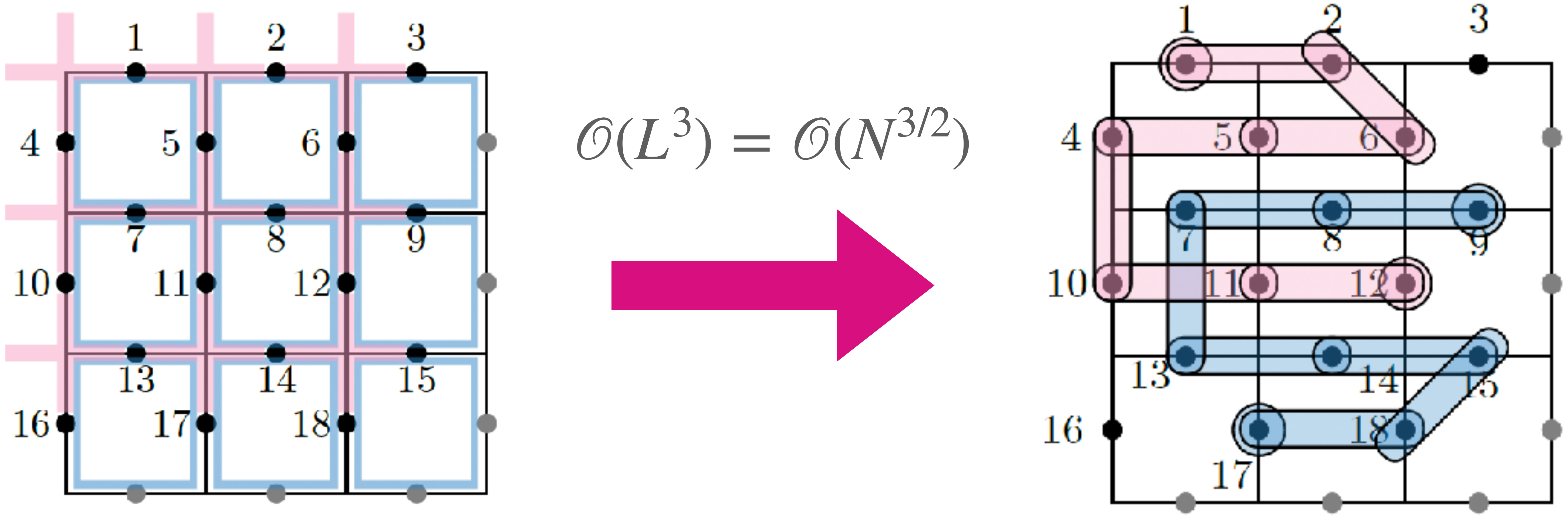

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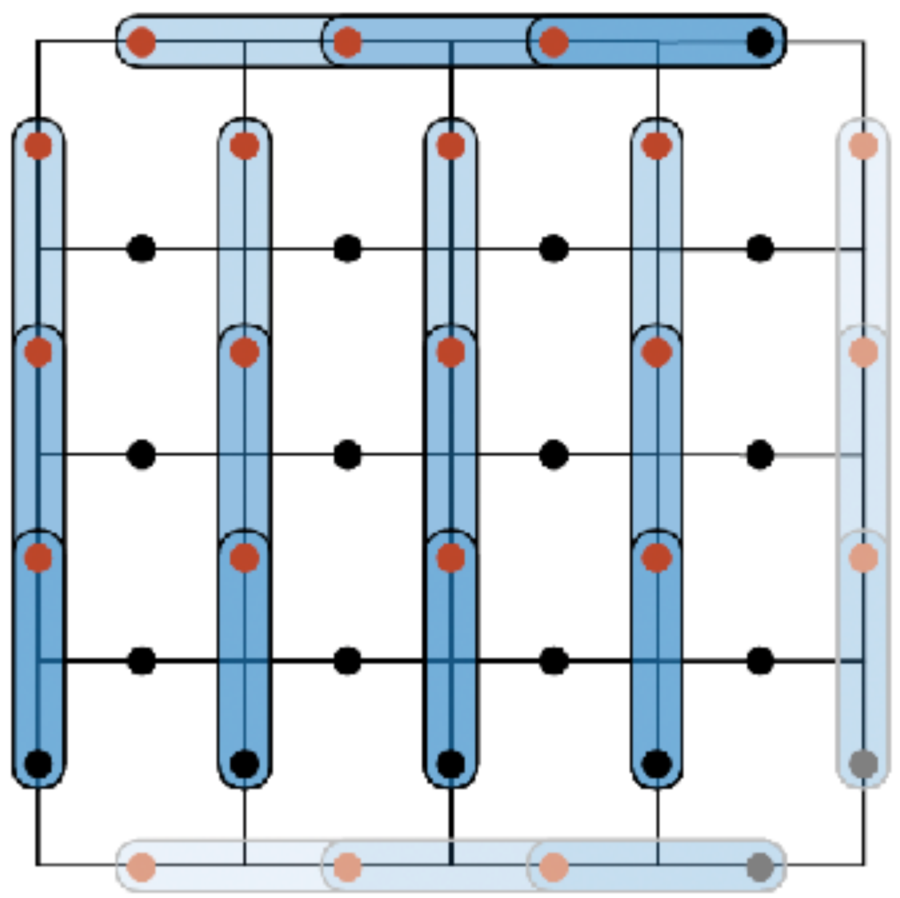
Some of the steps:

- Layer of Hadamard gates
- CX gates



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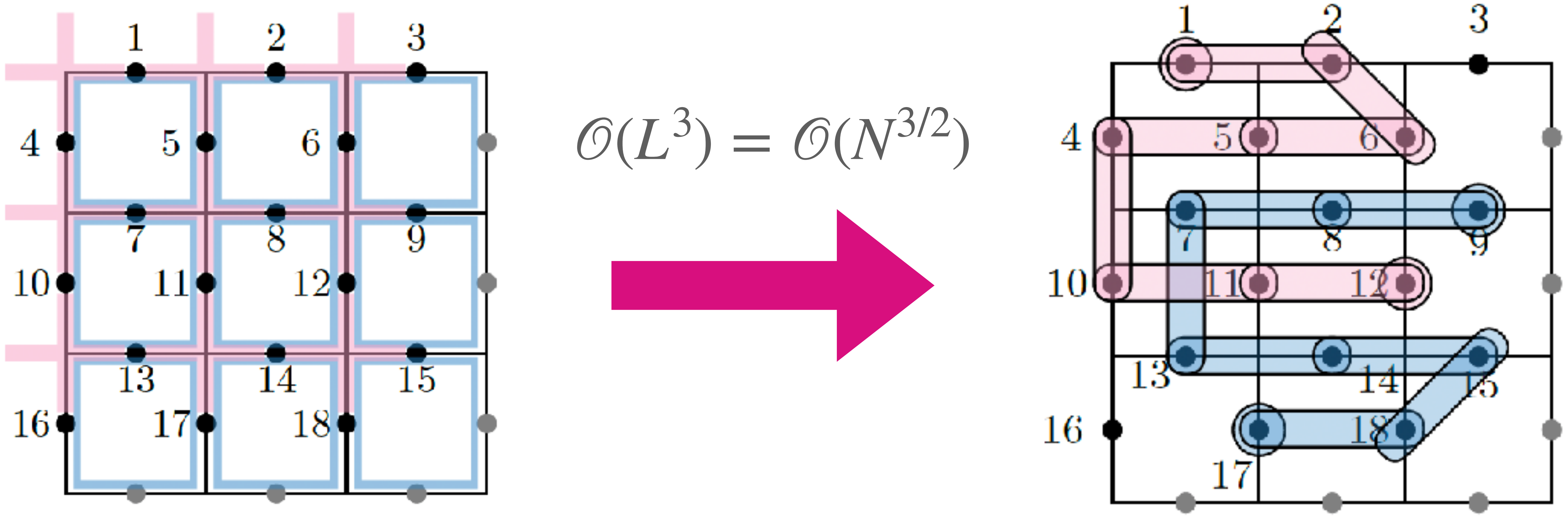

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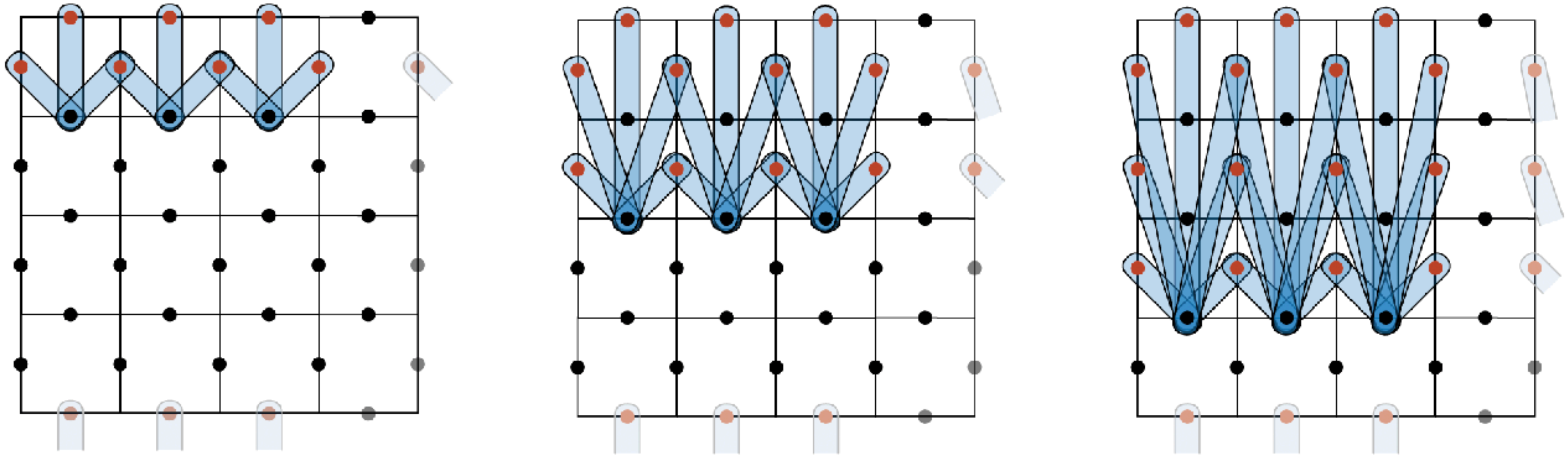

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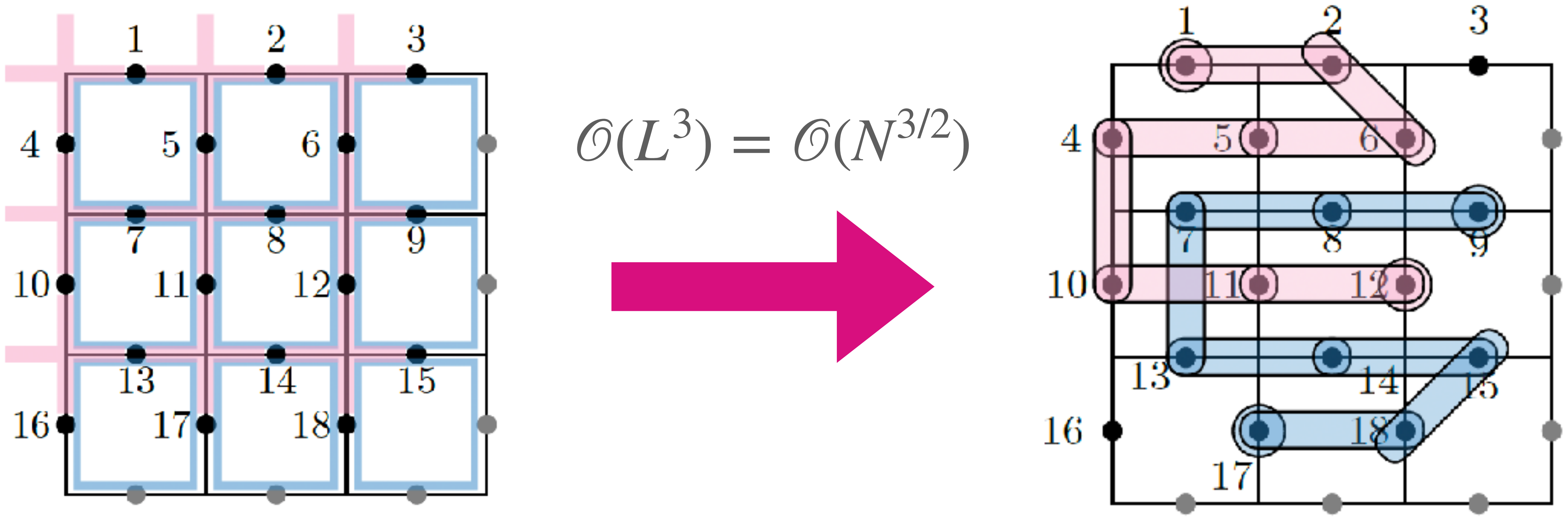

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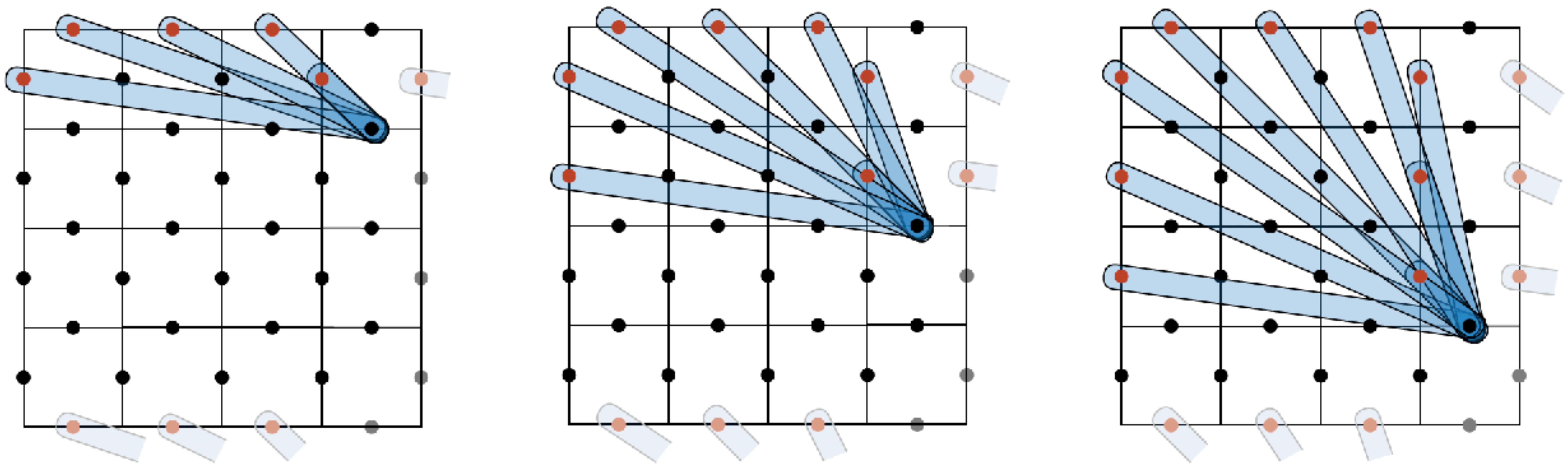

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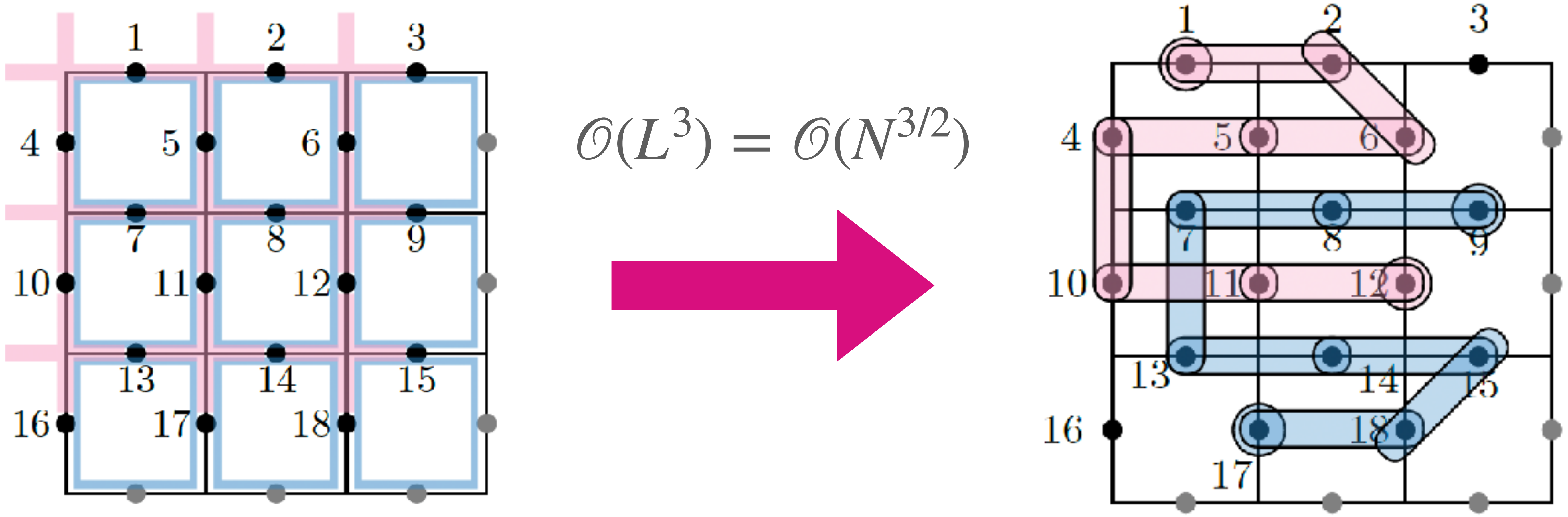
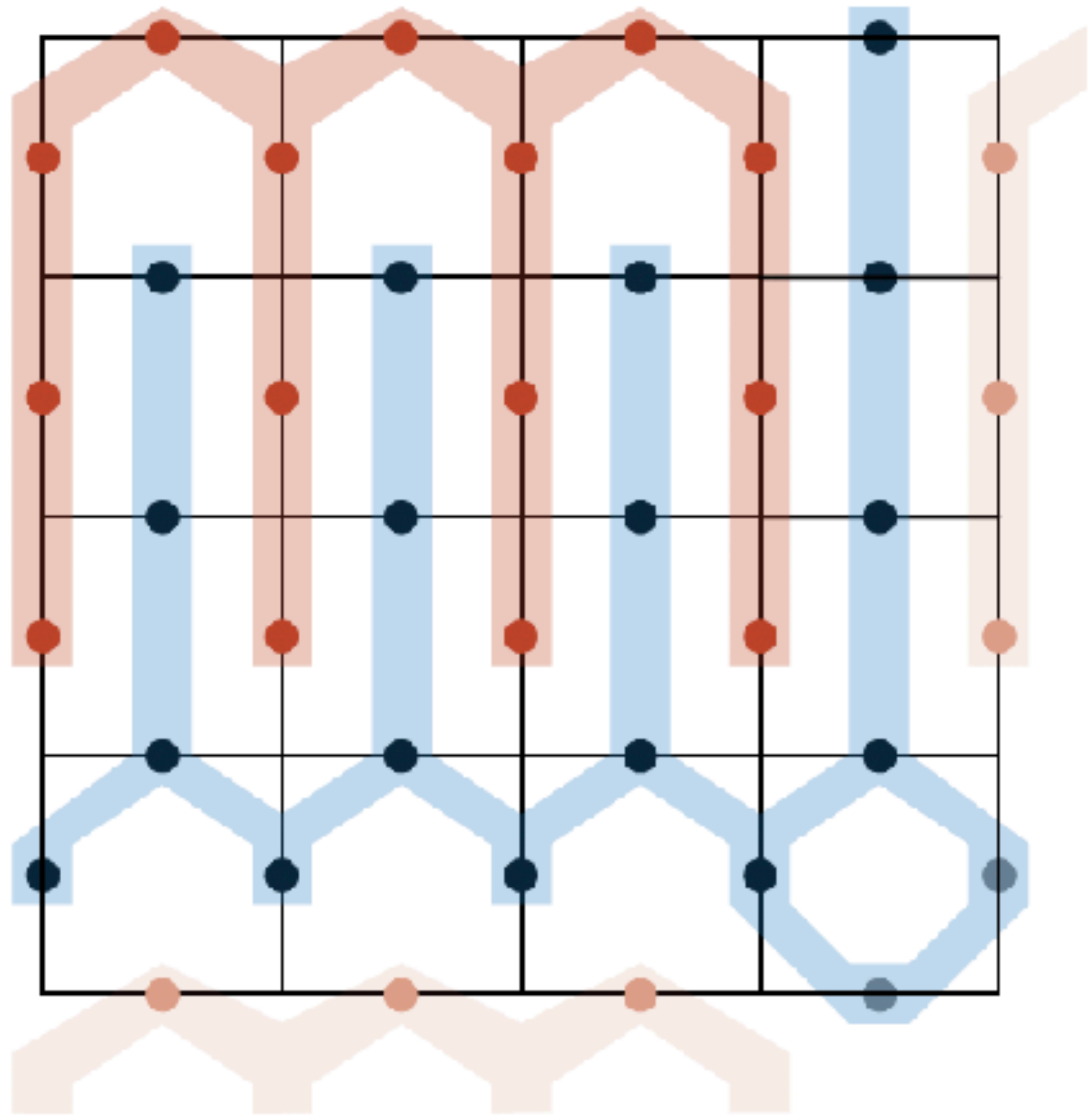


After this, we have two decoupled systems:



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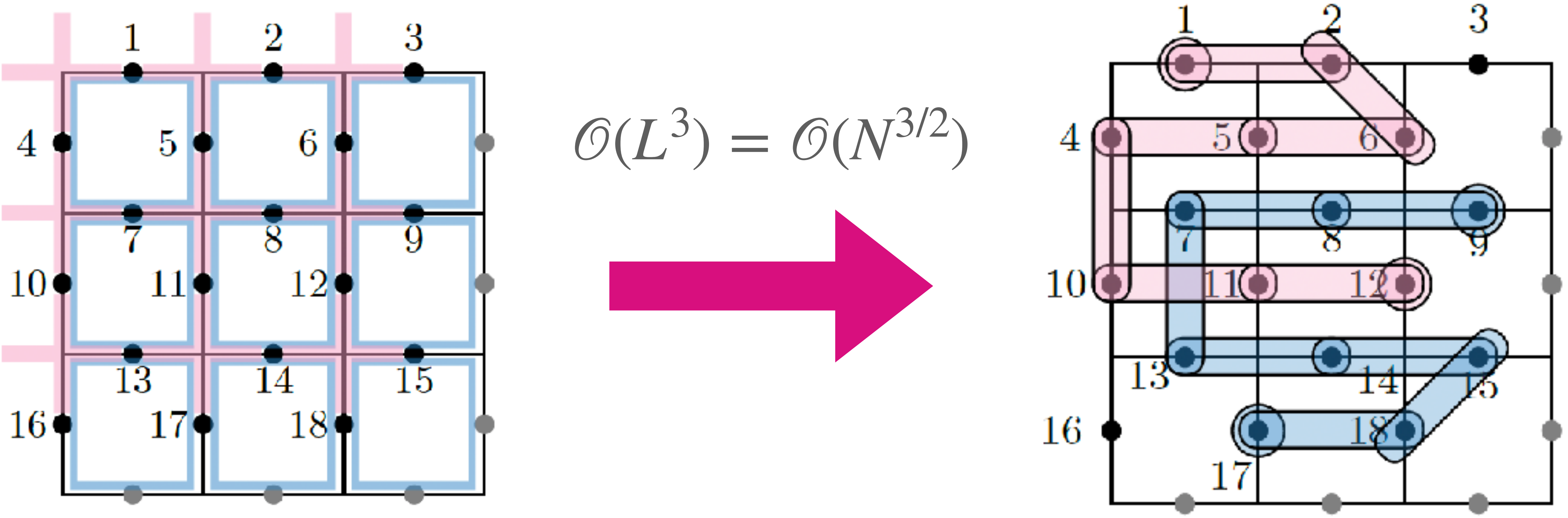
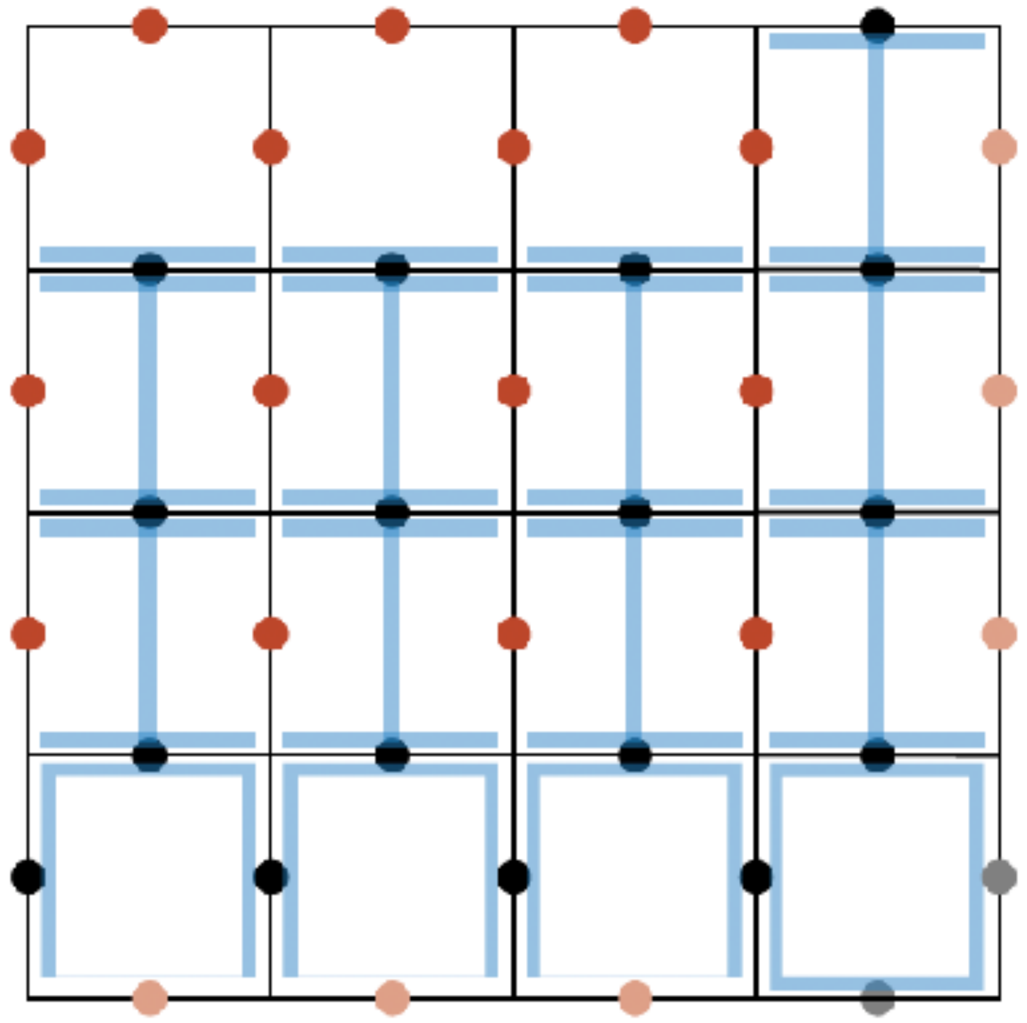
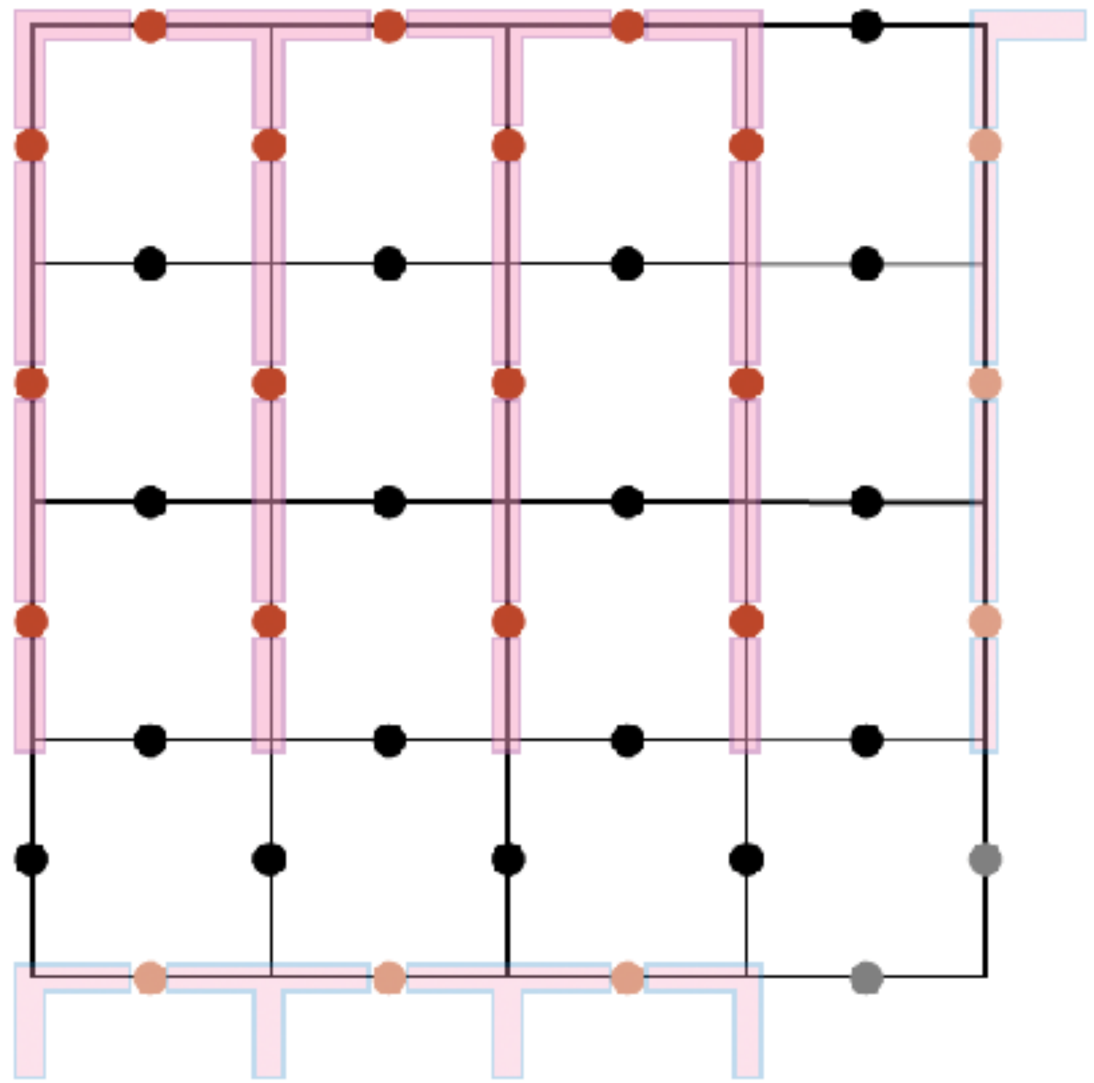


$\mathcal{O}(L^3) = \mathcal{O}(N^{3/2})$

Final plaquette interactions:



Final star interactions:



# DUALITY BETWEEN TORIC CODE AND CLASSICAL ISING CHAINS

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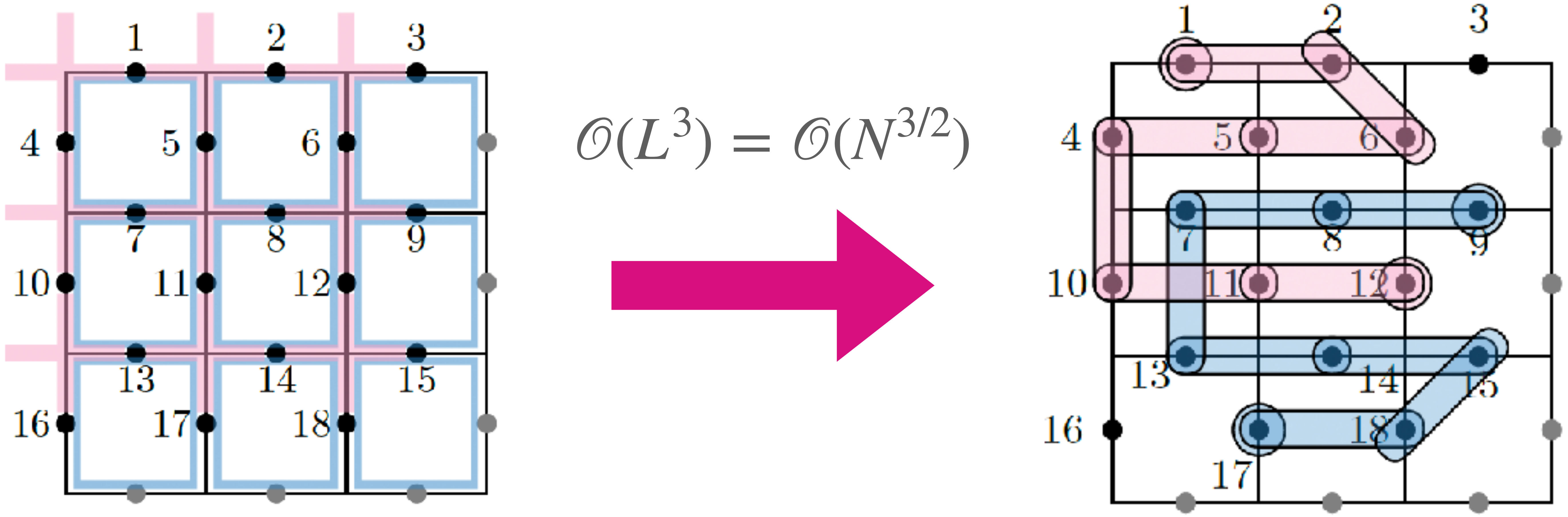
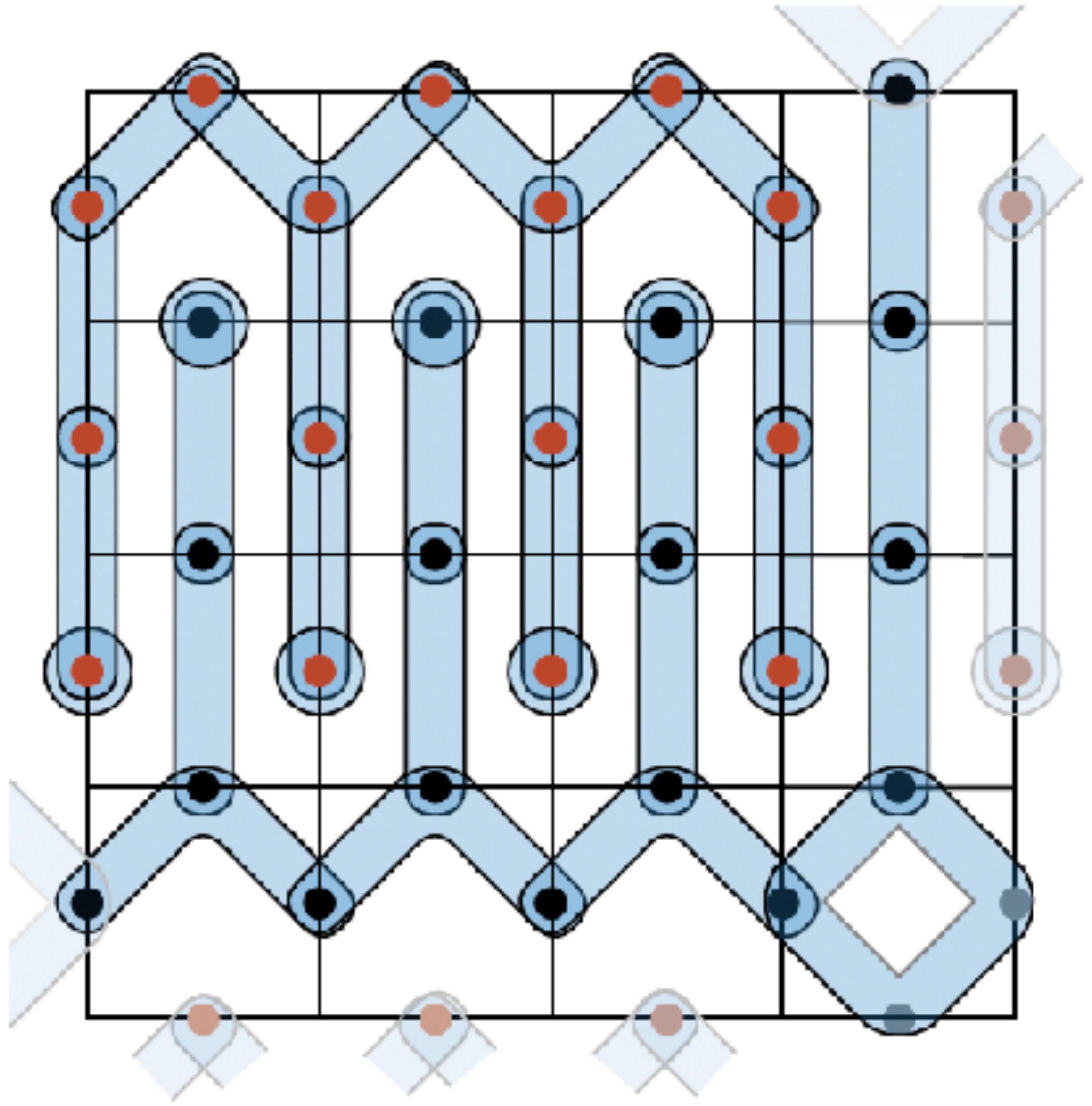


$\mathcal{O}(L^3) = \mathcal{O}(N^{3/2})$

Representation of the final interactions:



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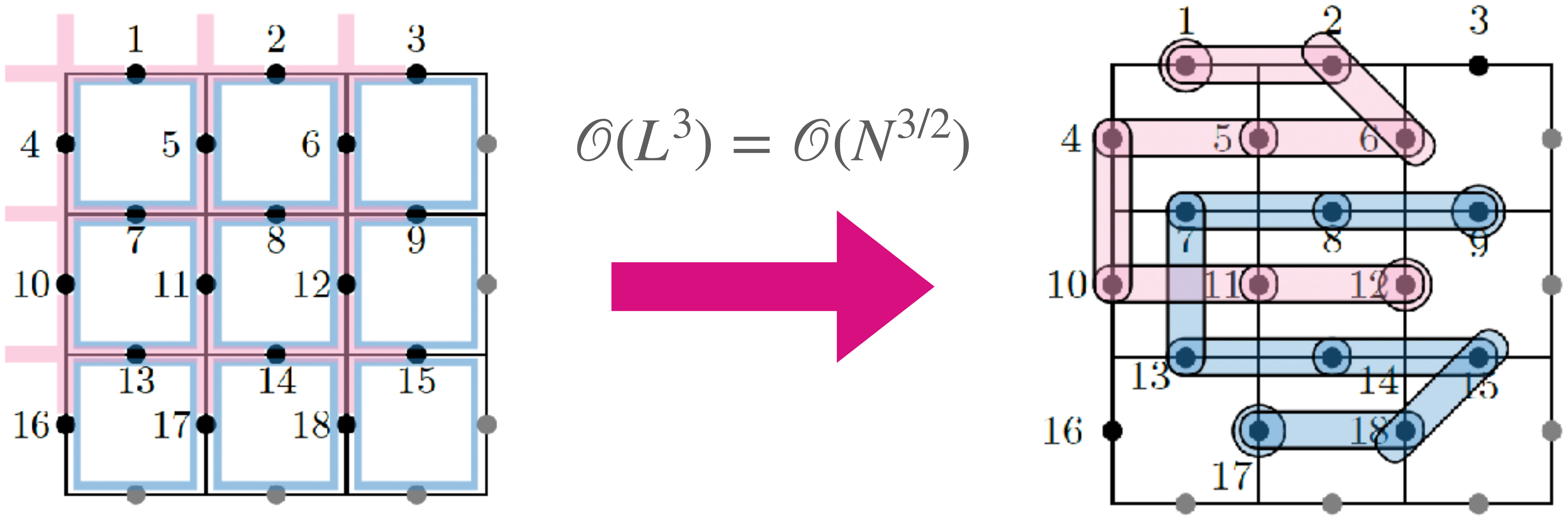

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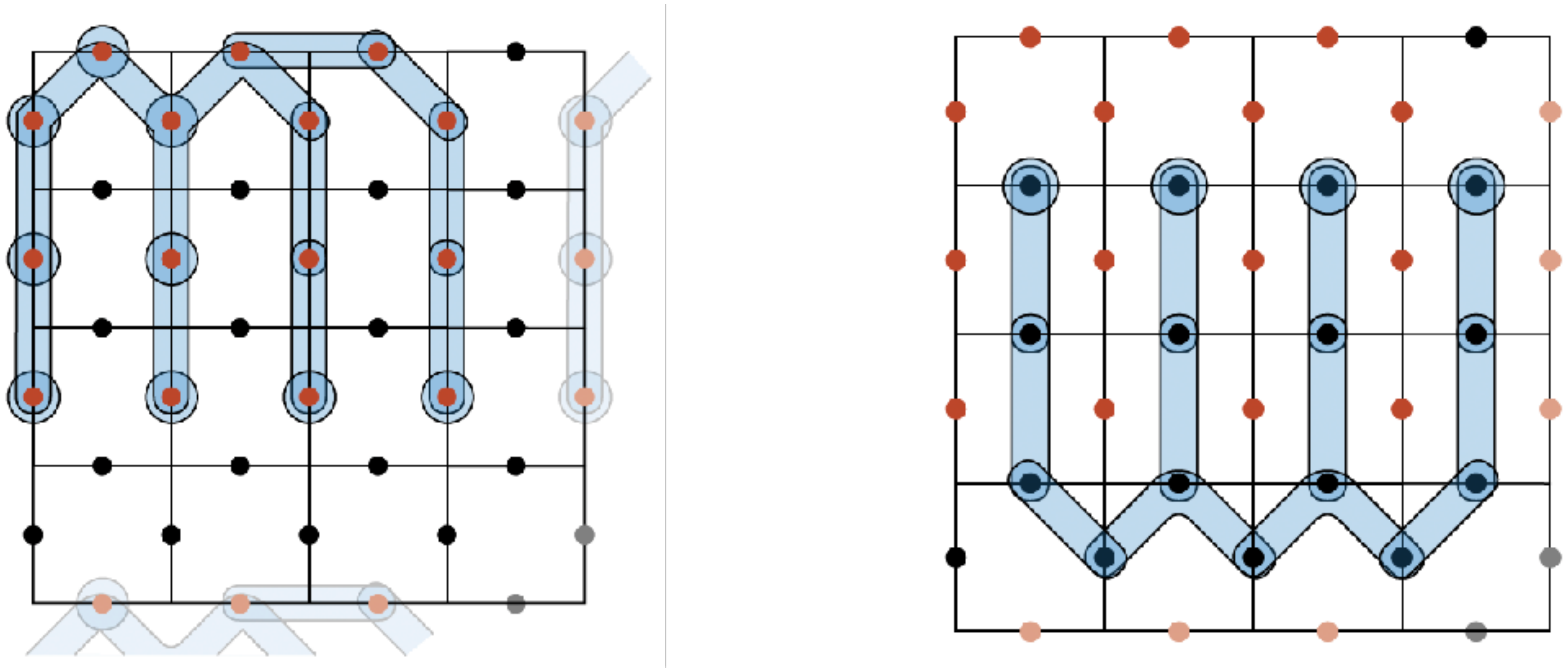
After some more $CX$ gates:



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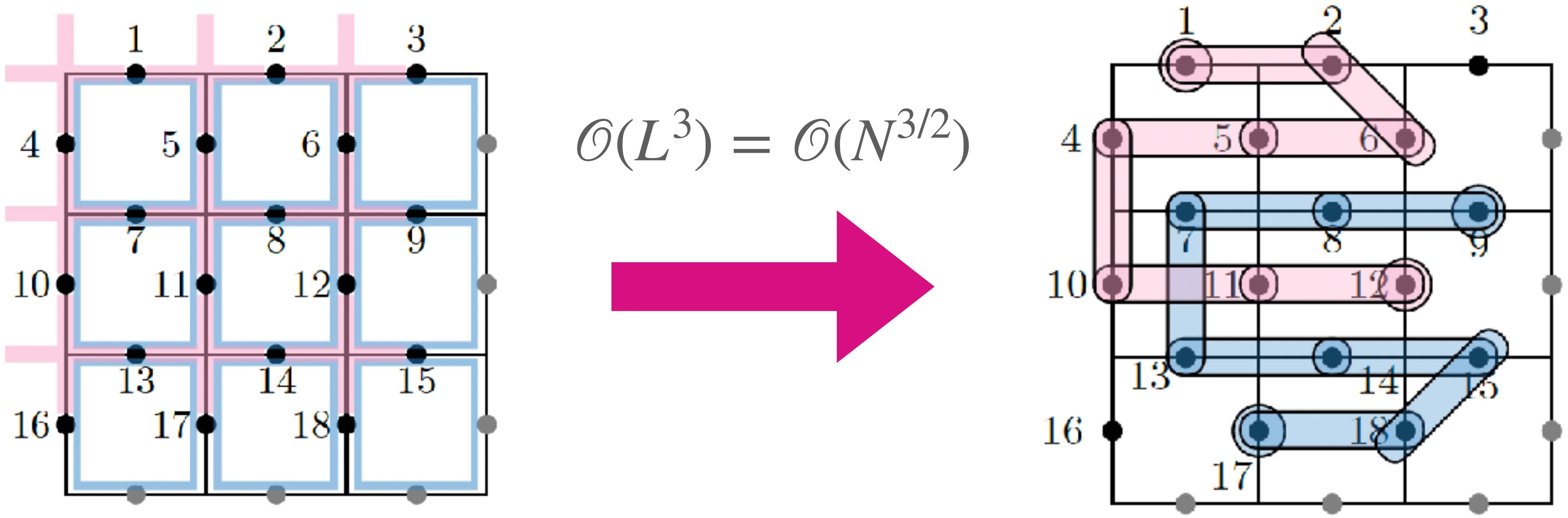

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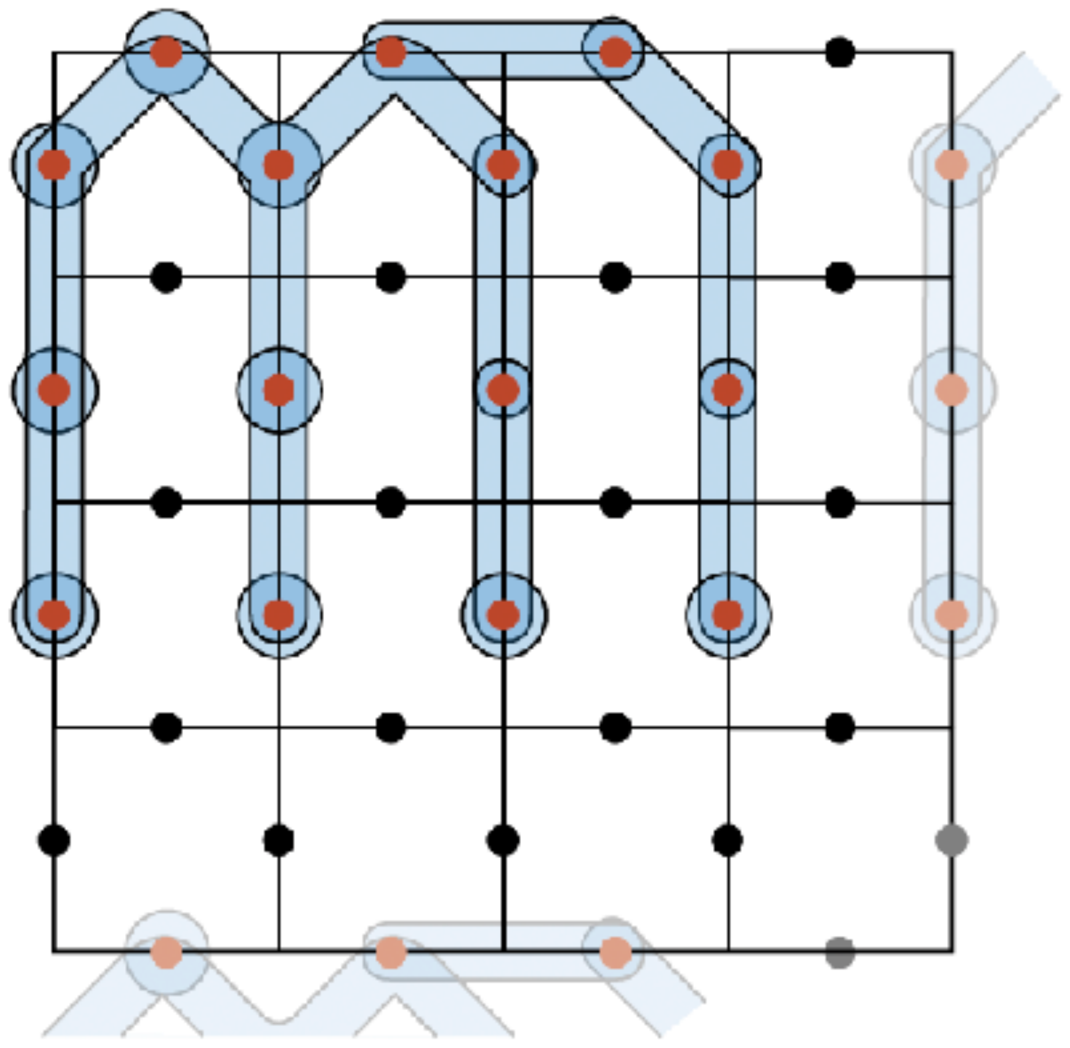
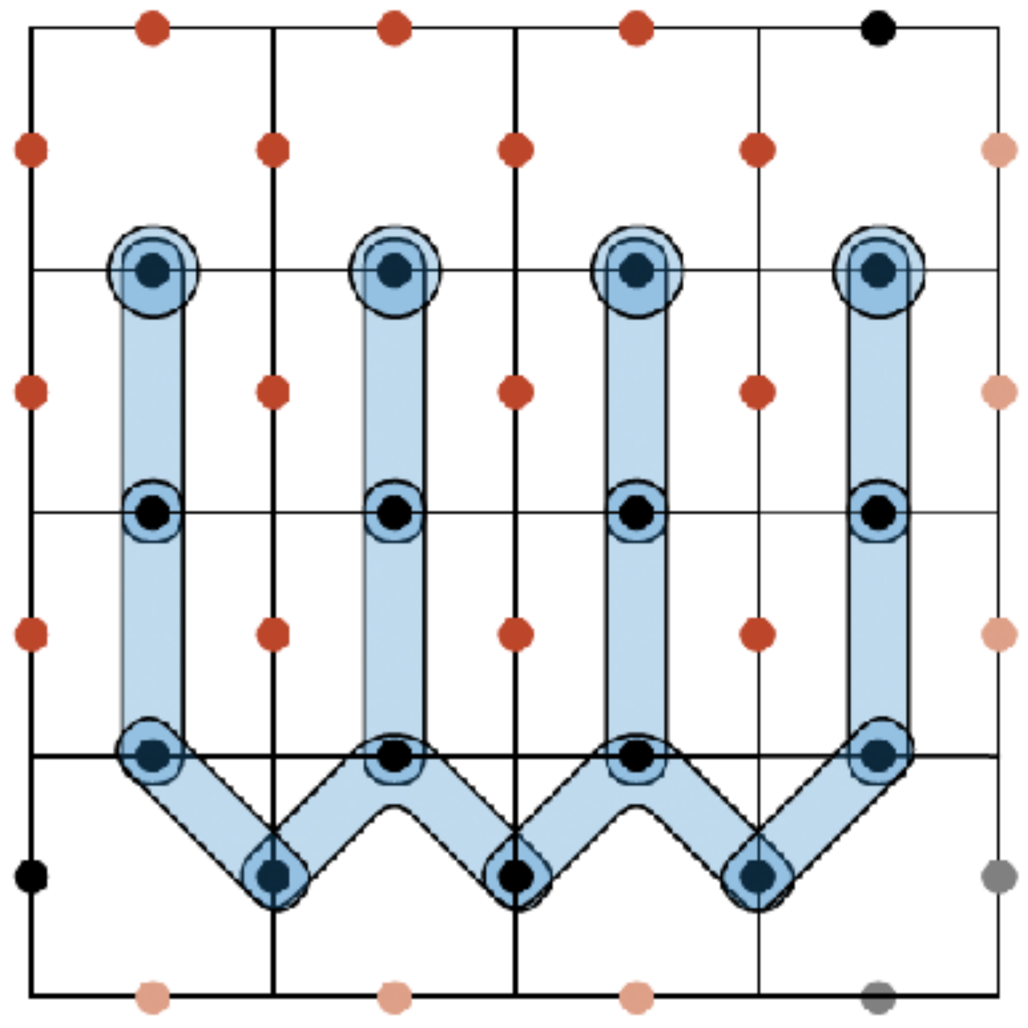
Final step: In each of these geometries, we get one interaction on all sites and magnetic fields in all sites. This is easily mapped to 2 Ising chains.





# Duality between Toric code and Classical Ising chains

## MAIN RESULT

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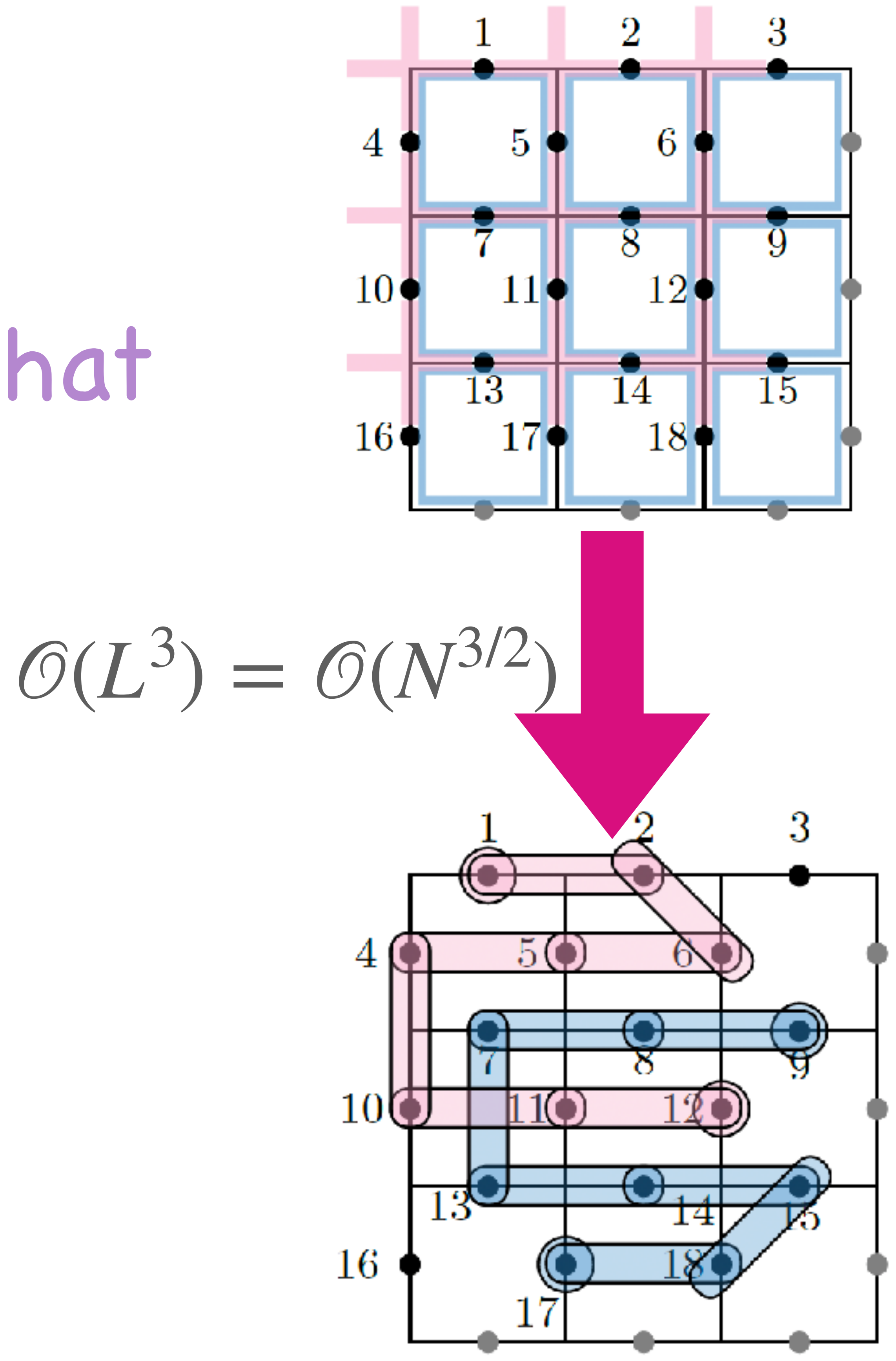

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# DUALITY OF OTHER CSS CODES

CSS CODE

Hamiltonian $-\sum_{v \in V_L} J_v A_v - \sum_{p \subset \mathcal{E}_L} J_p B_p$ $\qquad A_v := \bigotimes_{i \in \partial v} \sigma_x^i, \quad B_p := \bigotimes_{i \in p} \sigma_z^i.$

with more general geometries.

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## Commuting Pauli operators

$$H = \sum_{i=1}^{m} \alpha_i H_i.$$

with $\{H_i\}$ a collection of mutually orthogonal Pauli strings.

# DUALITY OF OTHER CSS CODES

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

## Result

The $\{H_i\}$ can be simultaneously diagonalised with a quantum circuit of cuadratic depth.

[van den Berg, Temme, Quantum'20]

[Aaronson, Gottesman, PRA'04]

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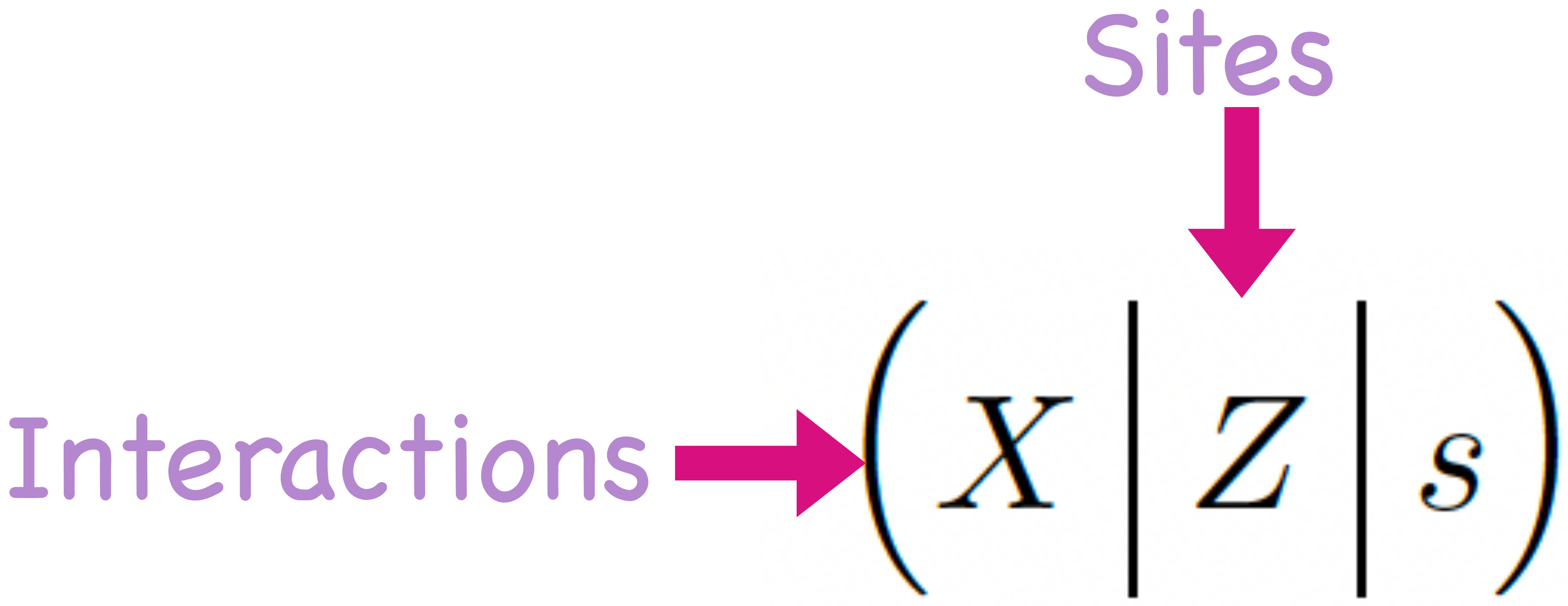
## Idea of the proof

Write interactions of the Hamiltonian in a **tableau:**

| Operator | $x_{ij}$ | $z_{ij}$ |
|---|---|---|
| $\sigma_x$ | 1 | 0 |
| $\sigma_z$ | 0 | 1 |
| $\sigma_y$ | 1 | 1 |
| $\mathbb{1}$ | 0 | 0 |



Sites

Interactions

$$\left( X \,\middle|\, Z \,\middle|\, s \right)$$

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Sites

Interactions → $\left( X \,\middle|\, Z \,\middle|\, s \right)$

Example

$$\sigma_z \otimes \sigma_y \otimes \mathbb{1} - \sigma_x \otimes \mathbb{1} \otimes \sigma_y \rightarrow \left(\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

# DUALITY OF OTHER CSS CODES

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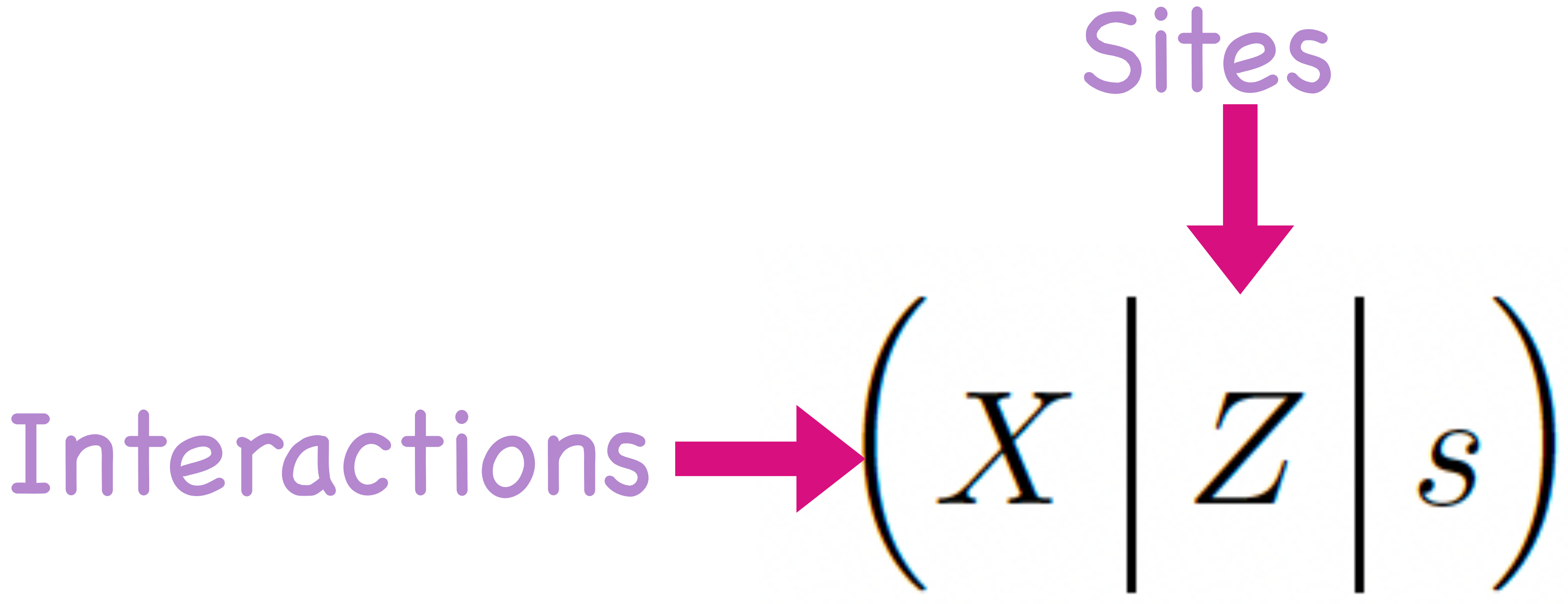

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Then, the aim is to reduce the $X$ part of the matrix to all 0s and analyse the remaining $Z$ part.

# DUALITY OF OTHER CSS CODES

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

## Result

The $\{H_i\}$ can be simultaneously diagonalised with a quantum circuit of cuadratic depth.

## Idea of the proof

Write interactions of the Hamiltonian in a **tableau**:

Then, the aim is to reduce the $X$ part of the matrix to all 0s and analyse the remaining $Z$ part.

For these models, this is done with $CX$, Hadamard and Phase gates in $\mathcal{O}(n^2)$ depth.

# Duality of other CSS Codes

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

## Result

The $\{H_i\}$ can be simultaneously diagonalised with a quantum circuit of cuadratic depth.

These shows that all Hamiltonians composed of commuting Pauli operators are poly-depth dual to classical Hamiltonians.

Now the question is: To which classical Hamiltonians?

# DUALITY OF OTHER CSS CODES

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

Example

If a tableau is achieved with $Z$ part like

$$\left(\begin{array}{c|c|c} \mathbf{I} & \mathbf{0} & \begin{matrix}0\,0\\ \vdots\end{matrix} \\ \hline 1\cdots1 & 0\cdots0 & \vdots \\ \hline \mathbf{0} & \mathbf{I} & \vdots \\ \hline 0\cdots0 & 1\cdots1 & 0\,0 \end{array}\right)$$

these are two decoupled 1D Ising models and two spins without interactions.

# Duality of other CSS Codes

## Example

$$H = \sum_{i=1}^{m} \alpha_i H_i$$

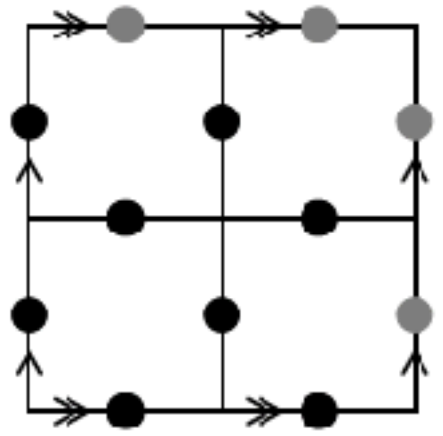
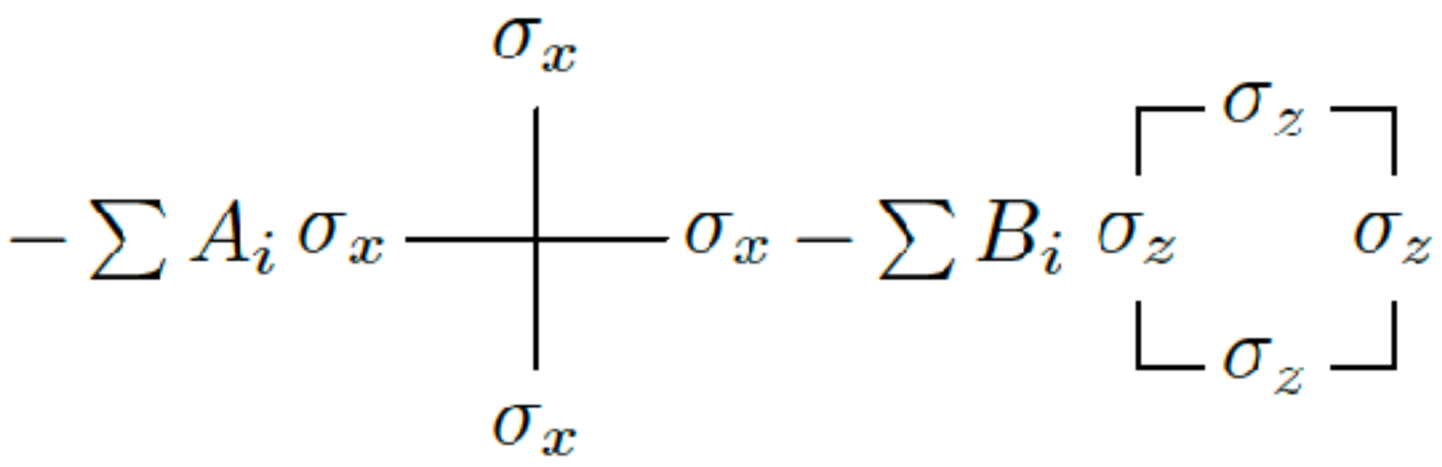
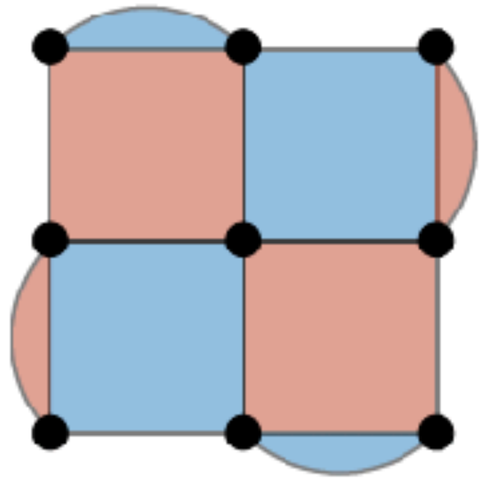
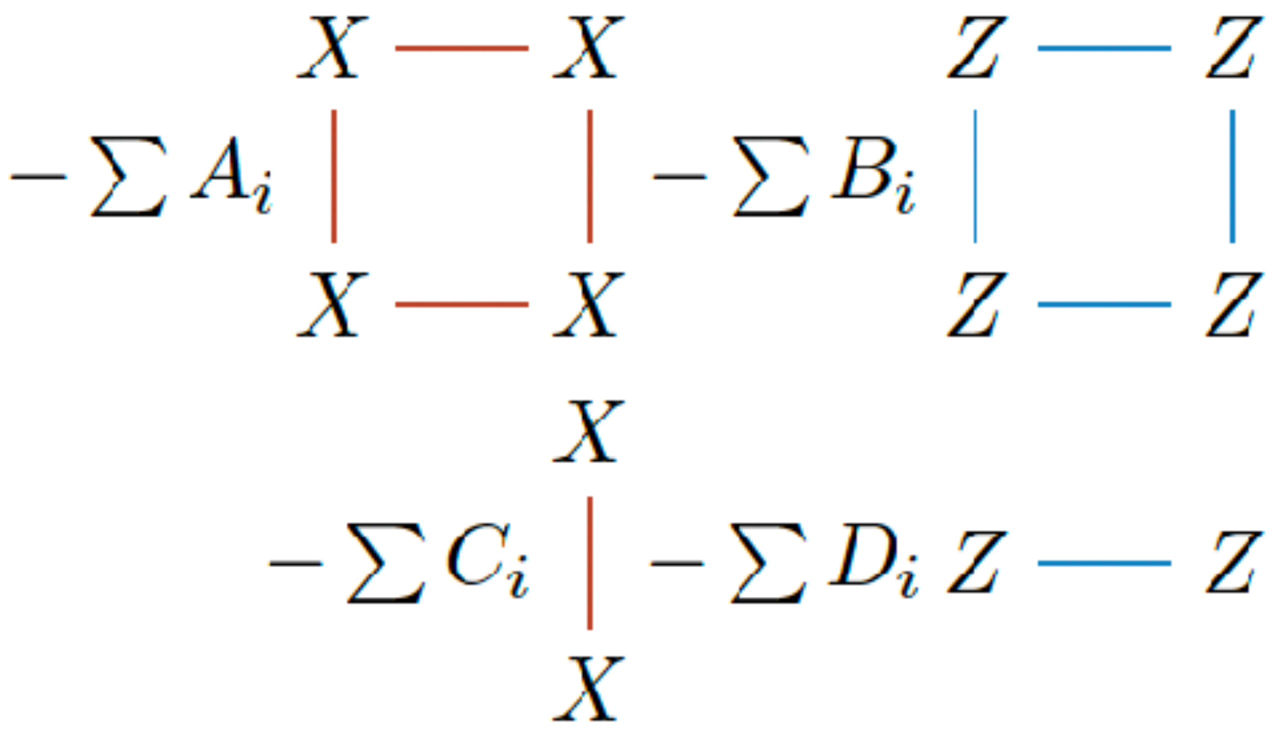
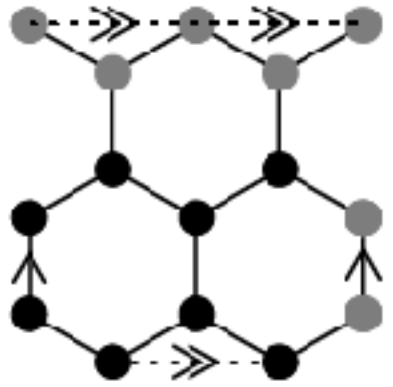
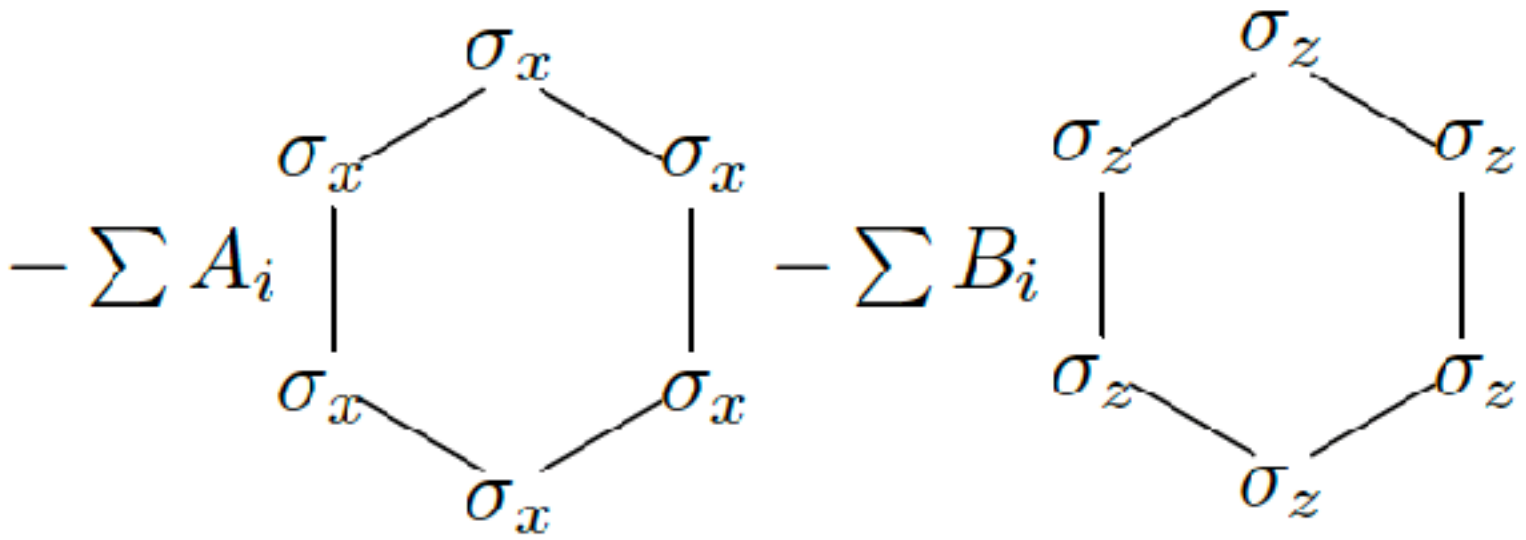
If a tableau is achieved with $Z$ part like

$$\left(\begin{array}{c|c|c} \mathbf{I} & \mathbf{0} & 0\,0 \\ & & \vdots \\ \cline{1-2} 1\cdots1 & 0\cdots0 & \vdots \\ \cline{1-2} \mathbf{0} & \mathbf{I} & \vdots \\ \cline{1-2} 0\cdots0 & 1\cdots1 & 0\,0 \end{array}\right)$$

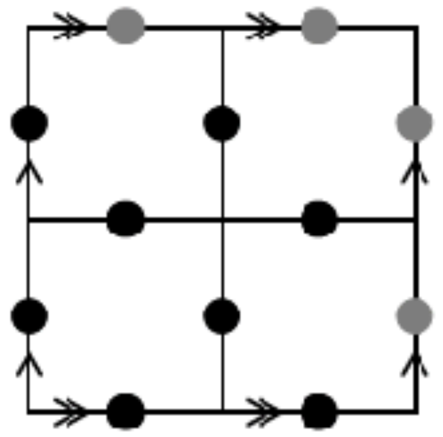
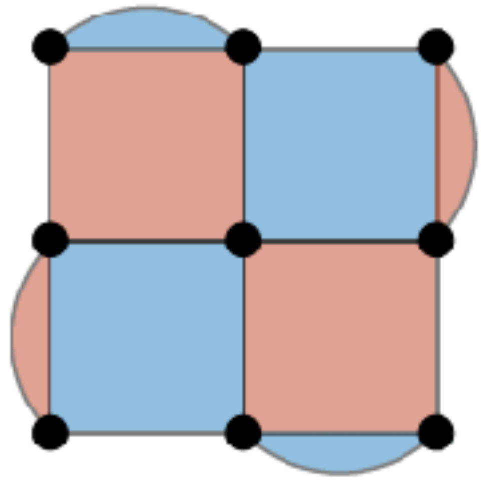
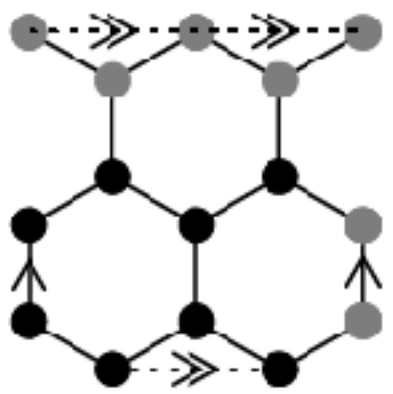
these are two decoupled 1D Ising models and two spins without interactions.

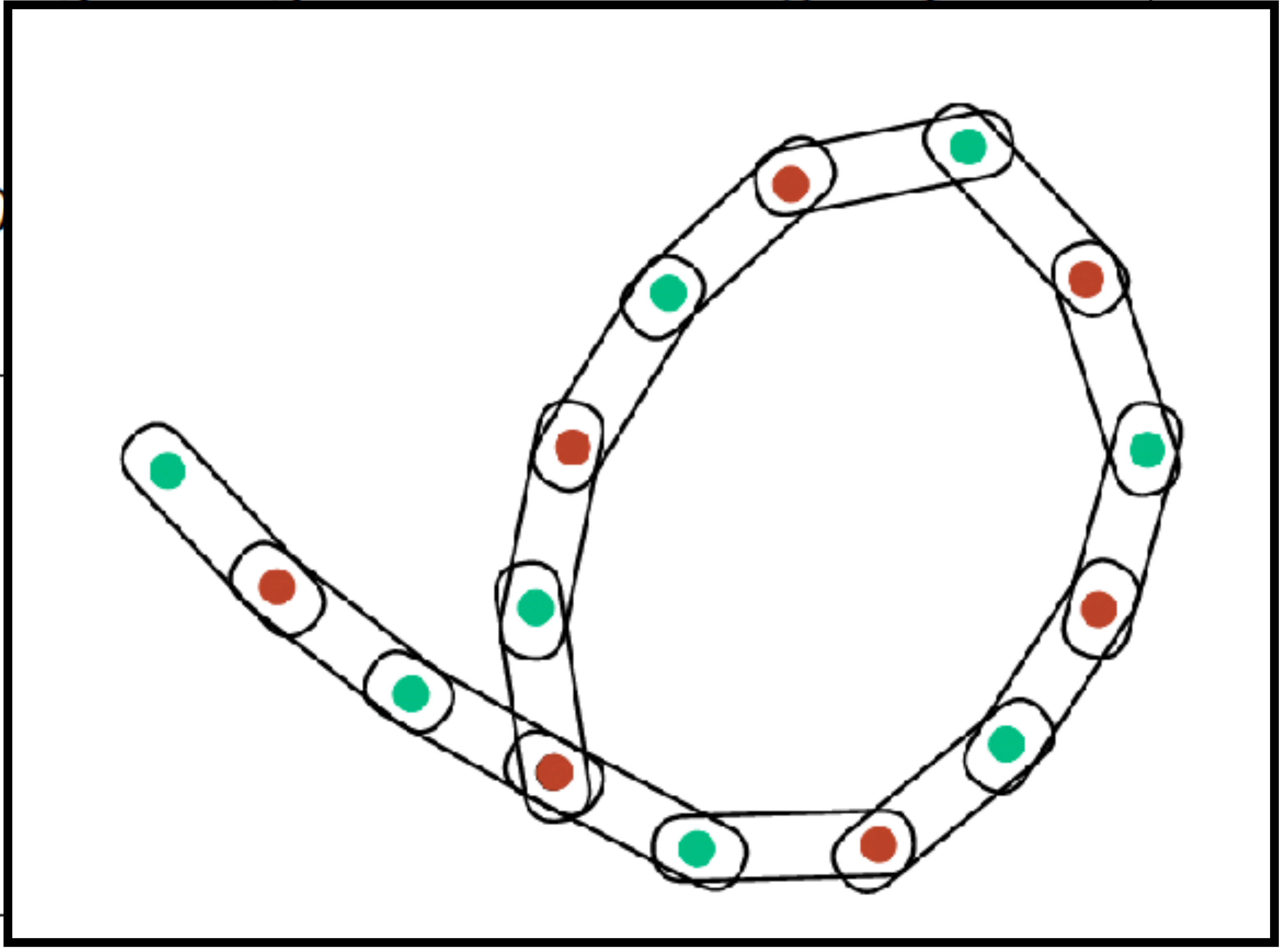
This is achieved from a 2D Toric Code.

# DUALITY OF OTHER CSS CODES

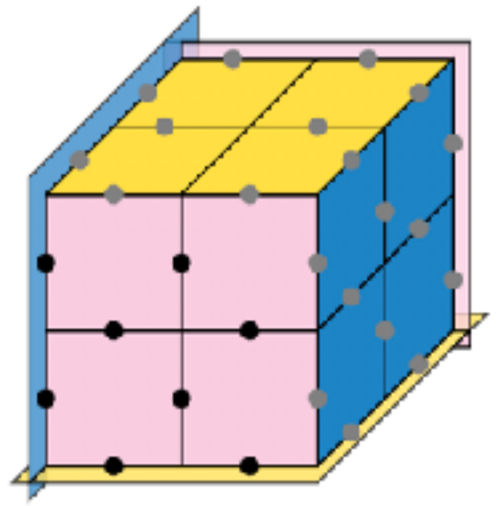
| **Original model** | **Lattice** | **Hamiltonian** | **Dual model** | |
|---|---|---|---|---|
| 2D toric code |  |  | Two decoupled Ising chains | Periodic boundary conditions |
| Rotated surface code |  |  | Non-interacting, single-spin Hamiltonian | Open boundary conditions |
| 2D color code on a honeycomb lattice |  |  | Two decoupled lasso Ising chains if or non-interacting, single-spin Hamiltonian. | Periodic boundary conditions |

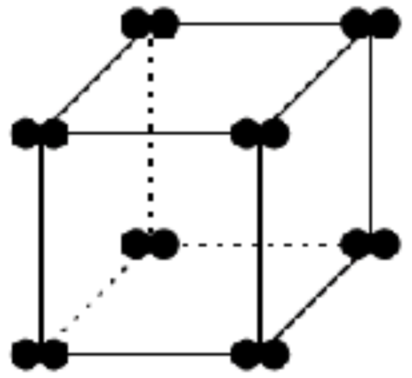
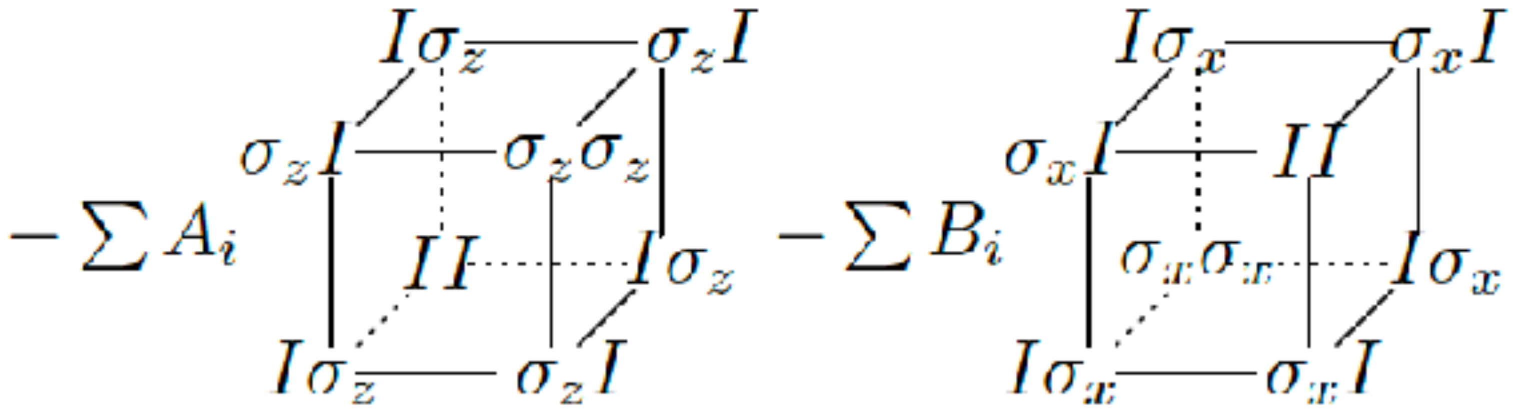
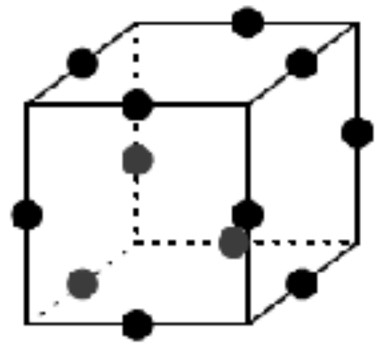
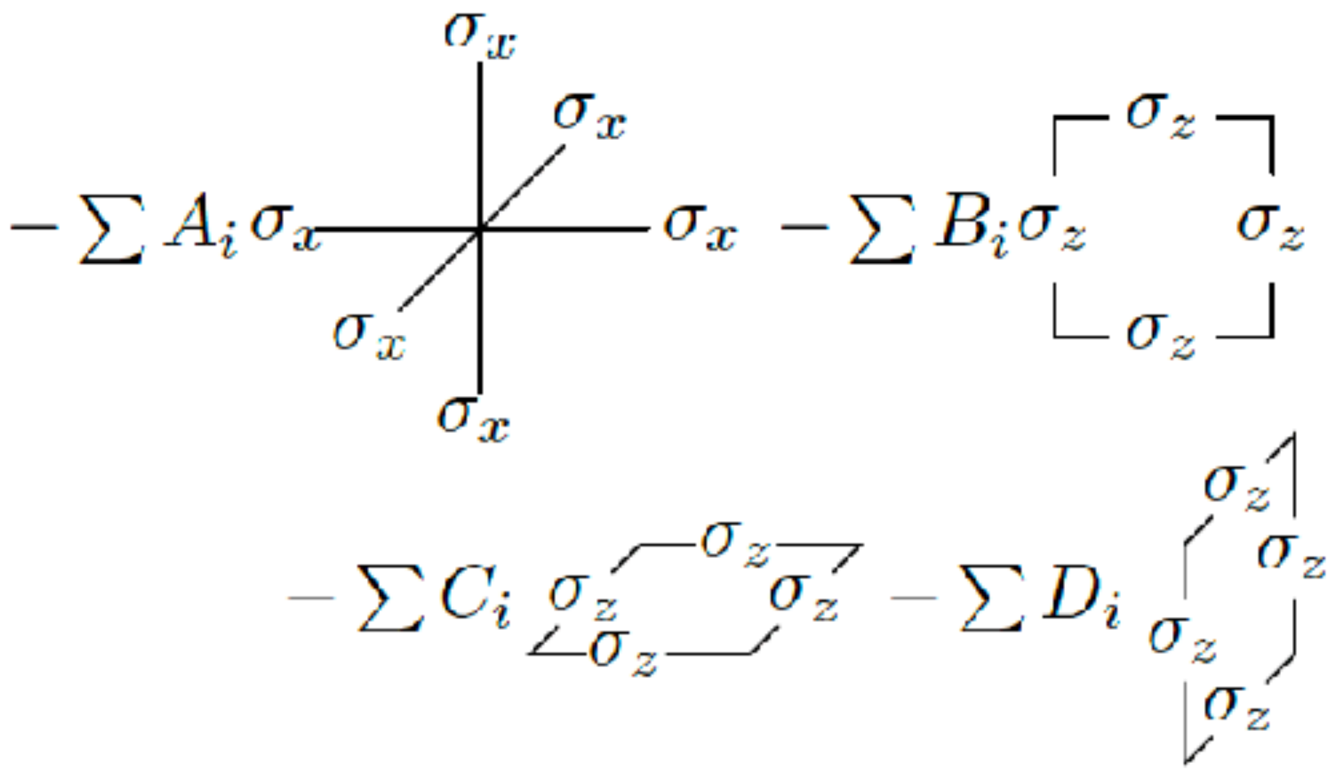
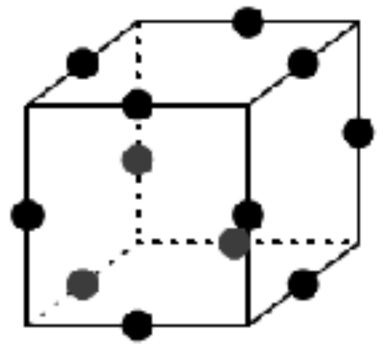
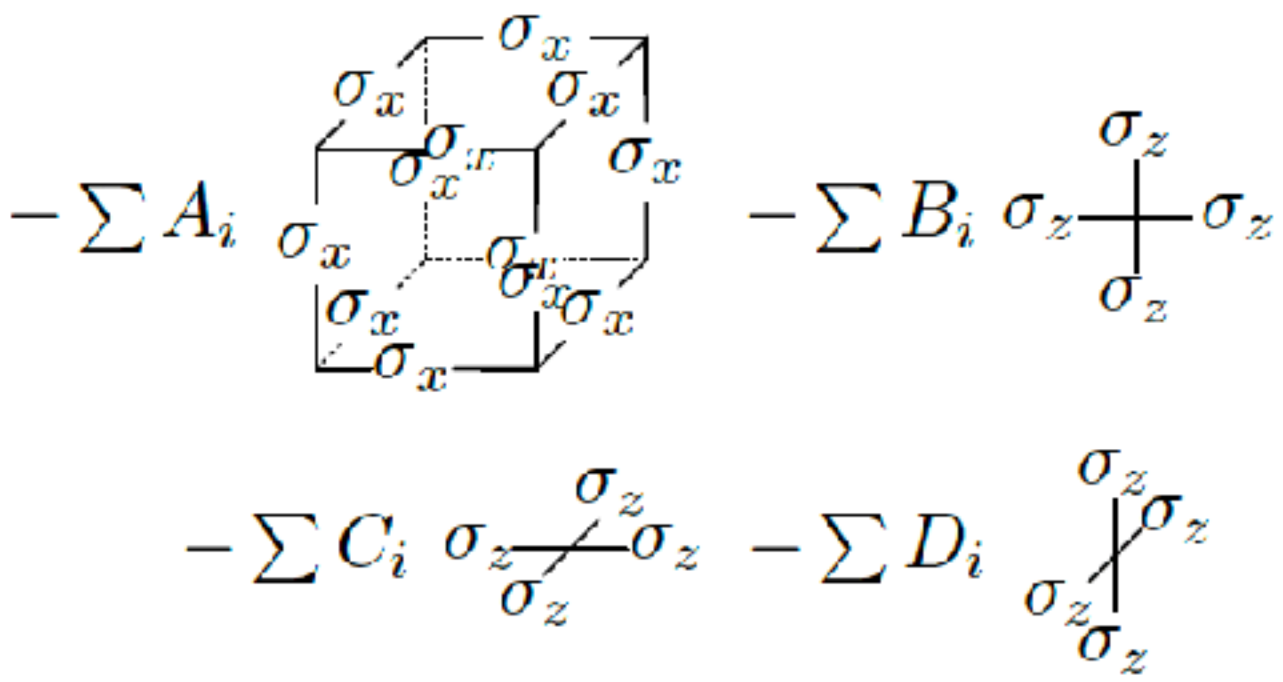
# DUALITY OF OTHER CSS CODES

| **Original model** | **Lattice** | **Hamiltonian** | **Dual model** |
|---|---|---|---|
| 2D toric code |  | $-\sum A_i \begin{array}{ccc} & \sigma_x & \\ \sigma_x & & \sigma_x \\ & \sigma_x & \end{array} -\sum B_i \begin{array}{ccc} & \sigma_z & \\ \sigma_z & & \sigma_z \\ & \sigma_z & \end{array}$ | Two decoupled Ising chains |
| Rotated surface code |  | $-\sum A_i \begin{array}{cc} X & X \\ X & X \end{array} -\sum B_i \begin{array}{cc} Z & Z \\ & \end{array}$<br>$-\sum C_i \begin{array}{c} X \\ X \end{array} -\sum D$ | Non-interacting, single-spin |
| 2D color code on a honeycomb lattice |  | $-\sum A_i \begin{array}{ccc} & \sigma_x & \\ \sigma_x & & \sigma_x \\ \sigma_x & & \sigma_x \\ & \sigma_x & \end{array} -\sum B_i$ | |

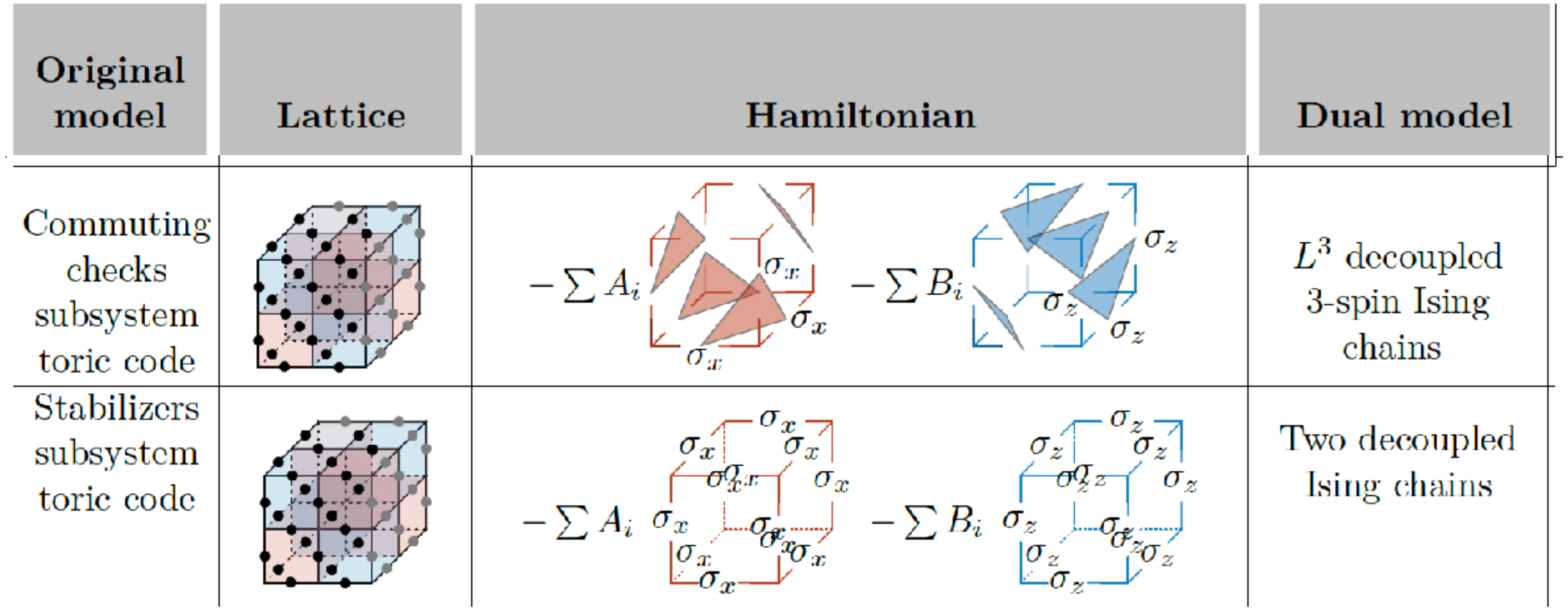


# DUALITY OF OTHER CSS CODES



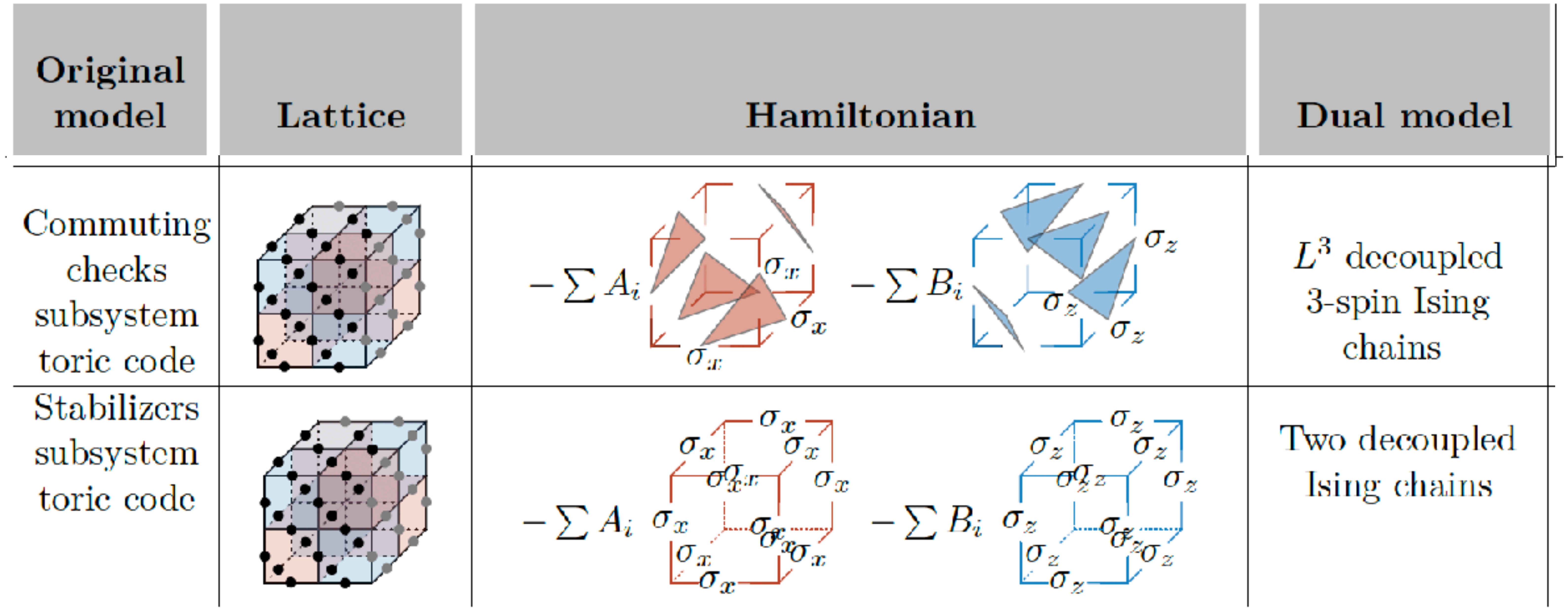
| Original model | Lattice | Hamiltonian | Dual model | |
|---|---|---|---|---|
| Haah's Code |  |  | Two decoupled Ising chains | Periodic boundary conditions |
| 3D toric code |  |  | Ising chain decoupled from a classical local model with constant degree interaction graph | Periodic boundary conditions |
| X-cube |  |  | $L$ decoupled Ising chains and $L-1$ 1D decoupled nearest-neighbor systems | Cylindrical boundary conditions |

# DUALITY OF OTHER CSS CODES



| Original model | Lattice | Hamiltonian | Dual model | |
|---|---|---|---|---|
| Commuting checks subsystem toric code | | $-\sum A_i$ $-\sum B_i$ | $L^3$ decoupled 3-spin Ising chains | Periodic boundary conditions |
| Stabilizers subsystem toric code | | $-\sum A_i$ $-\sum B_i$ | Two decoupled Ising chains | Periodic boundary conditions |

# DUALITY OF OTHER CSS CODES



| Original model | Lattice | Hamiltonian | Dual model |
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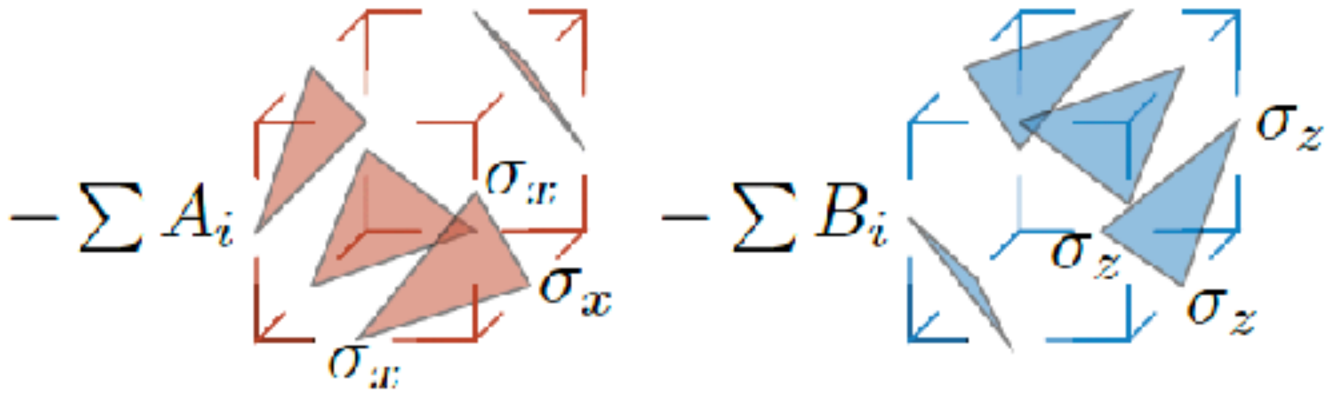
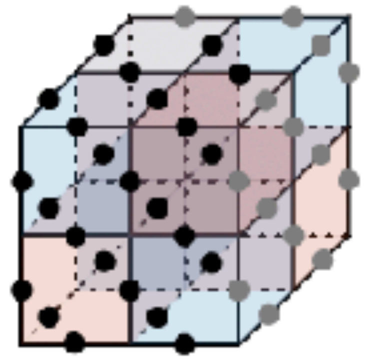
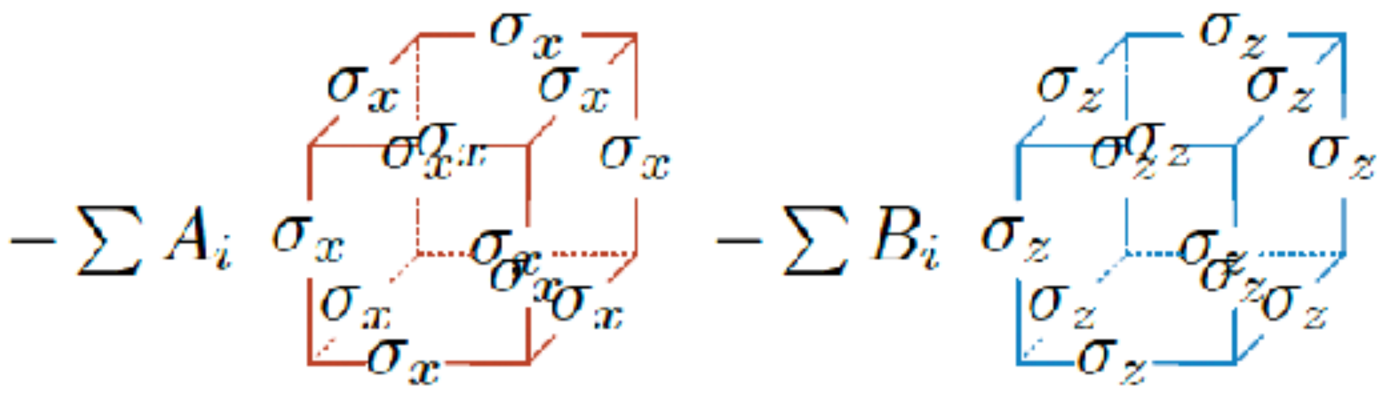
| Stabilizers subsystem toric code | | $-\sum A_i$ $-\sum B_i$ | Two decoupled Ising chains |

Periodic boundary conditions

Periodic boundary conditions

This is proven algorithmically for system sizes of order up to $10^5$ qubits and conjectured in general.

# DUALITY OF OTHER CSS CODES

| Original model | Lattice | Hamiltonian | Dual model | |
|---|---|---|---|---|
| Commuting checks subsystem toric code |  |  | $L^3$ decoupled 3-spin Ising chains | Periodic boundary conditions |
| Stabilizers subsystem toric code |  |  | Two decoupled Ising chains | Periodic boundary conditions |

Consequence: All these models can be efficiently sampled for any $0 < \beta \leq \infty$, except for the 3D toric code, for which we only have efficient sampling at $0 < \beta \leq \beta_*$ .

# DUALITY OF LINDBLADIANS

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Lindbladian $$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left[ L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]$$

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Then:

- If $\sigma$ is the unique fixed point of $\mathscr{L}$, $\widetilde{\sigma} = U\sigma U^\dagger$ is the unique fixed point of $\widetilde{\mathscr{L}}$.
- The spectral gap, MLSI and mixing time of $\mathscr{L}$ coincide with those of $\widetilde{\mathscr{L}}$.

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$$\begin{aligned} \|e^{t\mathscr{L}}(\sigma) - \rho\|_1 = \| \mathrm{Ad}_U \circ e^{t\mathscr{L}}(\sigma) - U\rho U^\dagger\|_1 &= \| \mathrm{Ad}_U \circ e^{t\mathscr{L}} \circ \mathrm{Ad}_{U^\dagger}(U\sigma U^\dagger) - \widetilde{\rho}\|_1 \\ &= \|e^{t\widetilde{\mathscr{L}}}(\widetilde{\sigma}) - \widetilde{\rho}\|_1 \end{aligned}$$

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$$\sup_{\sigma \in \mathcal{S}(\mathscr{H})} \|e^{t\mathscr{L}}(\sigma) - \rho\|_1 = \sup_{\sigma \in \mathcal{S}(\mathscr{H})} \|e^{t\widetilde{\mathscr{L}}}(\widetilde{\sigma}) - \widetilde{\rho}\|_1$$

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**Lindbladian** $\mathcal{L}(\rho) = -i[H,\rho] + \sum_k \gamma_k \left[ L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]$

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- If $\rho$ is the unique fixed point of $\mathscr{L}$, $\widetilde{\rho} = U\rho U^\dagger$ is the unique fixed point of $\widetilde{\mathscr{L}}$.
- The **spectral gap**, **MLSI** and **mixing time** of $\mathscr{L}$ coincide with those of $\widetilde{\mathscr{L}}$.

$$\sup_{\sigma\in\mathscr{S}(\mathscr{H})} \|e^{t\mathscr{L}}(\sigma) - \rho\|_1 = \sup_{\sigma\in\mathscr{S}(\mathscr{H})} \|e^{t\widetilde{\mathscr{L}}}(\widetilde{\sigma}) - \widetilde{\rho}\|_1$$

If $\sup_{\sigma\in\mathscr{S}(\mathscr{H})} \|e^{t\mathscr{L}}(\sigma) - \rho\|_1 < \varepsilon$ , then $\sup_{\sigma\in\mathscr{S}(\mathscr{H})} \|e^{t\widetilde{\mathscr{L}}}(\widetilde{\sigma}) - \widetilde{\rho}\|_1 < \varepsilon$

Mixing times coincide!

# DUALITY OF LINDBLADIANS

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Consider the dual Lindbladian $\widetilde{\mathcal{L}} := \mathrm{Ad}_U \circ \mathcal{L} \circ \mathrm{Ad}_{U^\dagger}$ with $\mathrm{Ad}_U(X) := UXU^\dagger$

- In particular, if $U$ is poly-depth and $\mathscr{L}$ is efficiently implementable, then $\widetilde{\mathscr{L}}$ also is!
- Note that this doesn't require $\widetilde{\mathscr{L}}$ to be local.

# CONCLUSIONS

- We have recalled **quantum Gibbs sampling via dissipation** and some systems for which it is efficient.
- We have introduced **quantum Gibbs sampling via duality.**
  - This has been used to show that the **2D toric code is dual** to **two 1D Ising chains,** for any system size.
  - Also algorithmically to show a computer-assisted proof of duality of other models of **commuting Pauli operators** to **classical Hamiltonians**, for small system sizes.
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THANKS FOR YOUR ATTENTION!