Mean-field dynamics and quantum fluctuations in long-range interacting open quantum systems

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Granada Summer School:
Quantum Matter Out of Equilibrium
1st - 5th September 2025

Benatti, FC, Floreanini, Narnhofer, JPA **50**, 423001 (2017); JPA **51**, 325001 (2018)
FC, Lesanovsky, PRL **126**, 230601 (2021), PRA **105**, L040202 (2022)
Boneberg, Lesanovsky, FC, PRA **106**, 012212 (2022)
Mattes, Lesanovsky, FC, PRA **108**, 062216 (2023)
FC, PRL **131**, 227102 (2023)
FC, Lesanovsky, PRL **133**, 150401 (2024)
Mattes, Lesanovsky, FC, PRL **134**, 070402 (2025)

Outline

Introduction

- Introduction
- Open quantum systems with long-range interactions

Mean-field dynamics in open quantum systems

- Collective observables and Heisenberg equations of motion
- Mean-field approximation: Proof of exactness for long-range models
- Time-crystal non-equilibrium phase transition

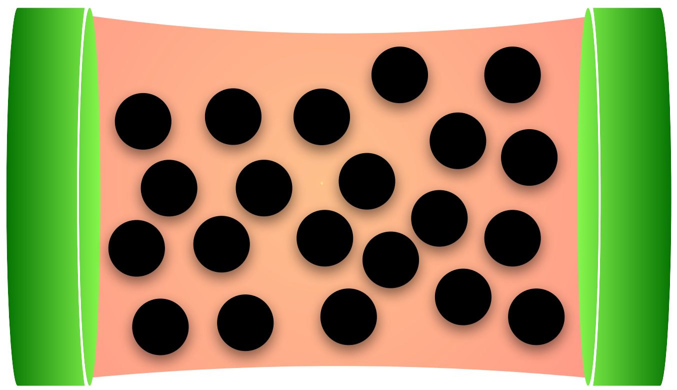
Quantum fluctuations and their dynamics

- How to account for correlations and their time evolution?
- Spin-squeezing in non-equilibrium stationary states
- Quantum and classical correlations in spin-boson models

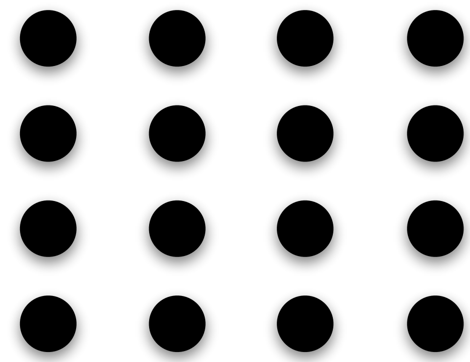
Algebraic approach

- Many-body spin systems

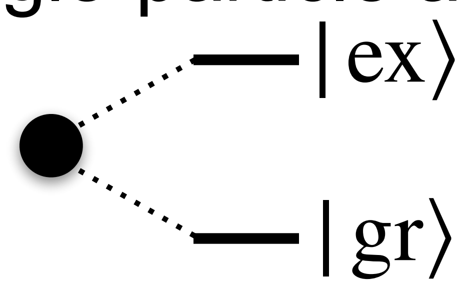
Cavity-atom systems



Atoms in a lattice



- Single-particle algebra



$$\{\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}}\} \leftrightarrow \mathscr{A}$$

Algebraic approach

- Many-body spin systems

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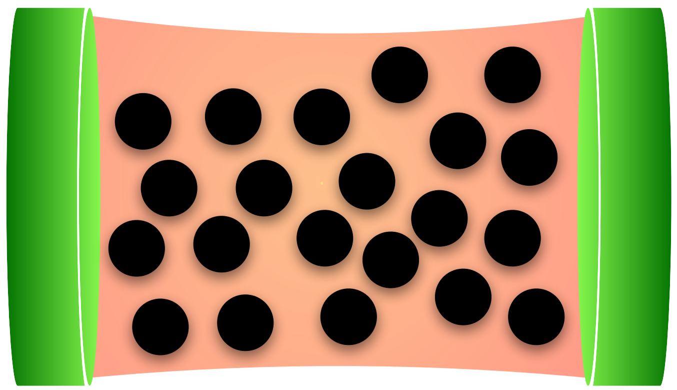
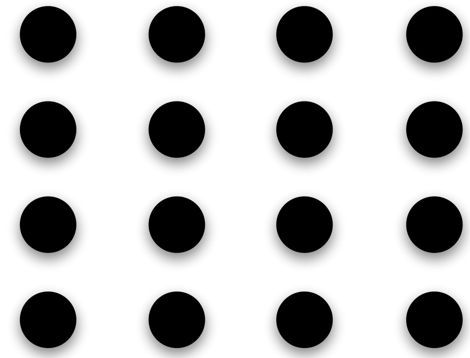
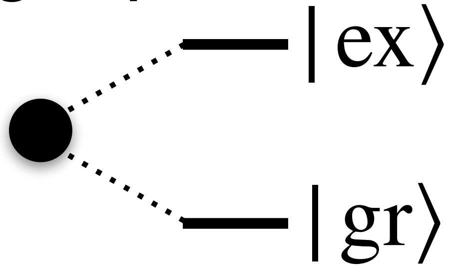


Atoms in a lattice



- Single-particle algebra



$$\{\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}}\} \leftrightarrow \mathscr{A}$$

- Many-particle algebra

$$\mathscr{A}_N = \bigotimes_{k=1}^{N} \mathscr{A}$$

$$\sigma_{\mathrm{r}}^{(k)} = \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes \sigma_{\mathrm{r}} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$$

k^{th} position

Quasi-local algebra

$$\mathscr{A}_\infty = \overline{\lim_{N \to \infty} \mathscr{A}_N}^{\|\cdot\|}$$

Example

$$O = \sum_{k=1}^{\infty} e^{-k} \sigma_{\mathrm{x}}^{(k)}$$

Algebraic approach

- Quantum states

Expectation values

$$\omega : \mathscr{A}_N \to \mathbb{C} \qquad \mathscr{A}_N \ni O \mapsto \langle O \rangle = \omega(O)$$

$$\begin{cases} \text{Positivity} & \omega(A^\dagger A) \geq 0 \\ \text{Normalisation} & \omega(\mathbf{1}) = 1 \\ \text{Linearity} & \omega(A + \lambda B) = \omega(A) + \lambda\omega(B) \end{cases}$$

Algebraic approach

- Quantum states

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- Additional properties

  - Translation invariance $\quad \omega\left(\sigma_{\mathrm{r}}^{(k)}\right) = \omega\left(\sigma_{\mathrm{r}}^{(h)}\right) = \omega(\sigma_{\mathrm{r}})$
  - Permutation invariance $\quad \omega\left(\sigma_{\mathrm{r}}^{(k_1)}\sigma_{\mathrm{s}}^{(h_1)}\right) = \omega\left(\sigma_{\mathrm{r}}^{(k_2)}\sigma_{\mathrm{s}}^{(h_2)}\right)$
  - Clustering $\quad \lim_{|k|\to\infty} \omega\left(\tau^k(A)B\right) = \omega(A)\omega(B)$

Example $\quad \omega\left(\sigma_{\mathrm{r}}^{(k)}\sigma_{\mathrm{s}}^{(h)}\right) = \omega(\sigma_{\mathrm{r}})\omega(\sigma_{\mathrm{s}}) \qquad k \neq h$

Product state

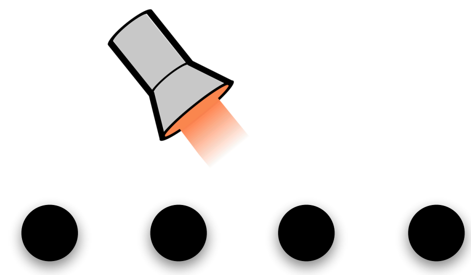
Hamiltonian operator

- Hamiltonian

$$\mathscr{A}_N \ni H_N \quad \mapsto \quad U_N(t) = e^{-itH_N}$$

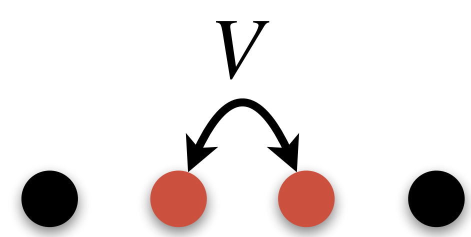
- Local fields

$$H_N \propto \Omega \sum_{k=1}^{N} \sigma_{\mathrm{x}}^{(k)}$$



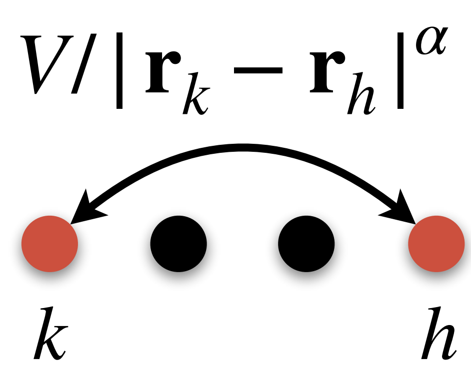
- Finite-range interactions

$$H_N \propto V \sum_{k=1}^{N} \sigma_{\mathrm{z}}^{(k)} \sigma_{\mathrm{z}}^{(k+1)}$$



- Power-law decaying interactions

$$H_N \propto V \sum_{k,h=1}^{N} \frac{\sigma_{\mathrm{z}}^{(k)} \sigma_{\mathrm{z}}^{(h)}}{|\mathbf{r}_k - \mathbf{r}_h|^{\alpha}}$$



$\mathbf{r}_k$ position vector for particle k

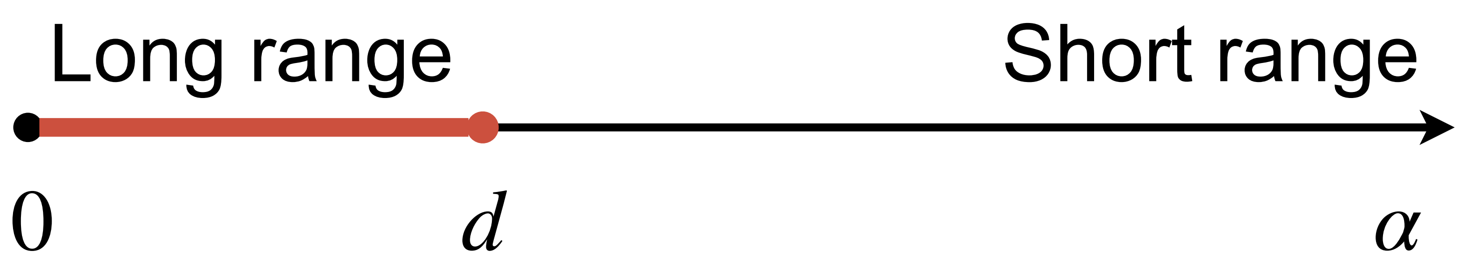
Short- vs long-range interactions

- Power-law exponent

$$H_N \propto V \sum_{k,h=1,k\neq h}^{N} \frac{\sigma_z^{(k)}\sigma_z^{(h)}}{|\mathbf{r}_k - \mathbf{r}_h|^{\alpha}}$$

$\alpha \rightarrow$ range of the interactions

$$H_N \propto V \sum_{k=1}^{N} \sum_{h\neq k}^{N} \frac{\sigma_z^{(k)}\sigma_z^{(h)}}{|\mathbf{r}_k - \mathbf{r}_h|^{\alpha}}$$



- Renormalisation: Kac-factor

$$H_N \propto \frac{V}{c_\alpha^N} \sum_{k,h=1,k\neq h}^{N} \frac{\sigma_z^{(k)}\sigma_z^{(h)}}{|\mathbf{r}_k - \mathbf{r}_h|^{\alpha}} \qquad c_\alpha^N \propto \begin{cases} N^{1-\alpha/d} & \alpha < d \\ \log N & \alpha = d \\ \text{const.} & \alpha > d \end{cases}$$

Defenu et al. RMP **95**, 035002 (2023)

Collective interactions

- All-to-all (collective) interactions with same strength

$$\alpha = 0$$

$$S_{\mathrm{r}} = \sum_{k=1}^{N} \sigma_{\mathrm{r}}^{(k)}$$

$$H_N \propto \frac{V}{N} \sum_{k,h=1}^{N} \sigma_{\mathrm{z}}^{(k)} \sigma_{\mathrm{z}}^{(h)} = \frac{V}{N} S_{\mathrm{z}} S_{\mathrm{z}}$$

- "Toy" models for equilibrium phase transitions
  - Curie-Weiss ferromagnet $H_N = -\frac{V}{N} S_{\mathrm{z}} S_{\mathrm{z}}$
  - BCS model $H_N = -\varepsilon S_{\mathrm{z}} - \frac{2T_{\mathrm{crit}}}{N} S_+ S_-$
    Thirring et al. CMP **4**, 303 (1967)
  - Spin-boson models (Dicke or Tavis-Cummings models)
    Hepp et al. Ann. Phys. **76**, 360 (1973)
    Kirton et al. Adv. Quantum Technol. 2, 1970013 (2019)

Collective interactions

- Efficient numerical simulation

$S^2 = S_x^2 + S_y^2 + S_z^2$ is conserved

$$S_z \,|\, S, m\rangle = m \,|\, S, m\rangle$$

$$S_+ \,|\, S, m\rangle \propto \,|\, S, m+1\rangle$$

Collective interactions

- Efficient numerical simulation

$$S^2 = S_x^2 + S_y^2 + S_z^2 \text{ is conserved}$$

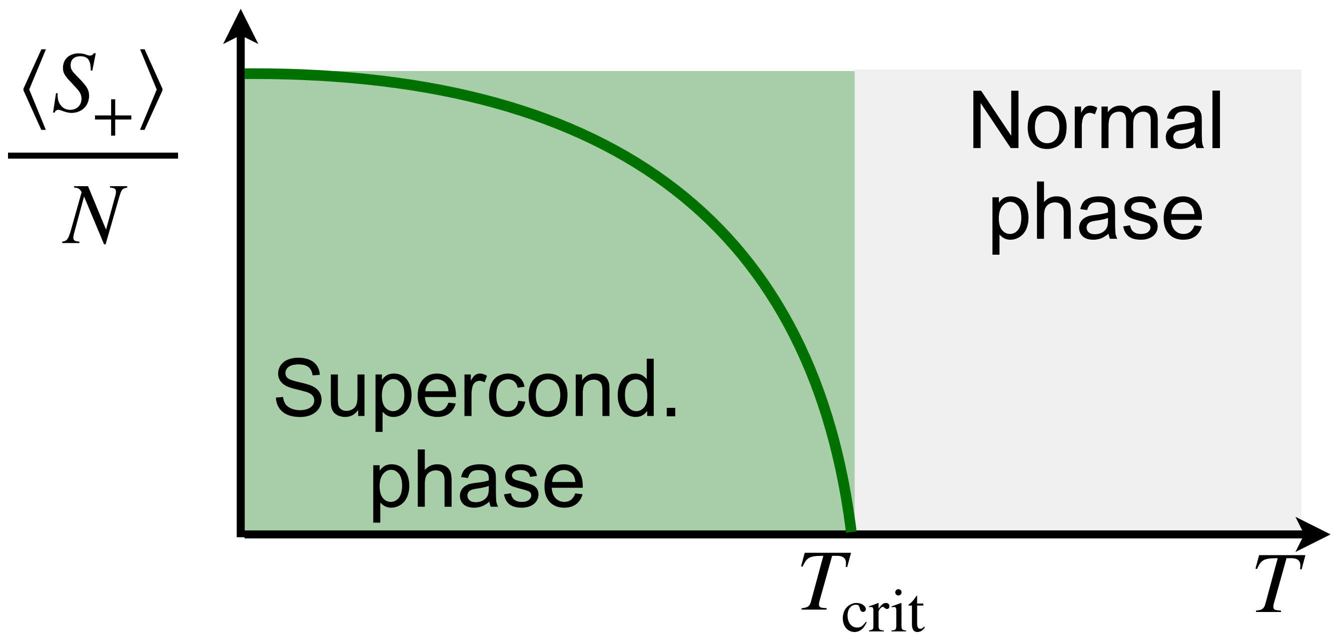
$$S_z |S, m\rangle = m |S, m\rangle$$
$$S_+ |S, m\rangle \propto |S, m+1\rangle$$

- Exact analytical results

$$H_N = -\varepsilon S_z - \frac{2T_{crit}}{N} S_+ S_-$$
$$\downarrow$$
$$h_N = -\varepsilon S_z - 2T_{crit} \left(\frac{\langle S_+ \rangle}{N} S_- + \frac{\langle S_- \rangle}{N} S_+ \right)$$

Self-consistency equations

$$\langle S_r \rangle = \frac{\mathrm{Tr}\left(e^{-\beta h_N} S_r\right)}{\mathrm{Tr}\left(e^{-\beta h_N}\right)}$$

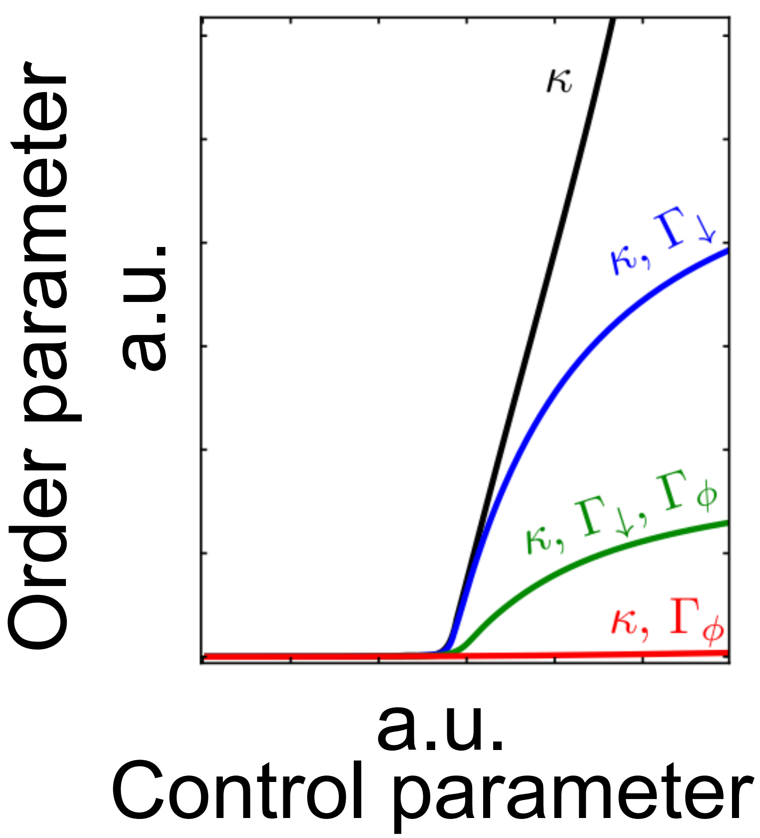


Thirring et al. CMP **4**, 303 (1967)

Why open quantum systems?

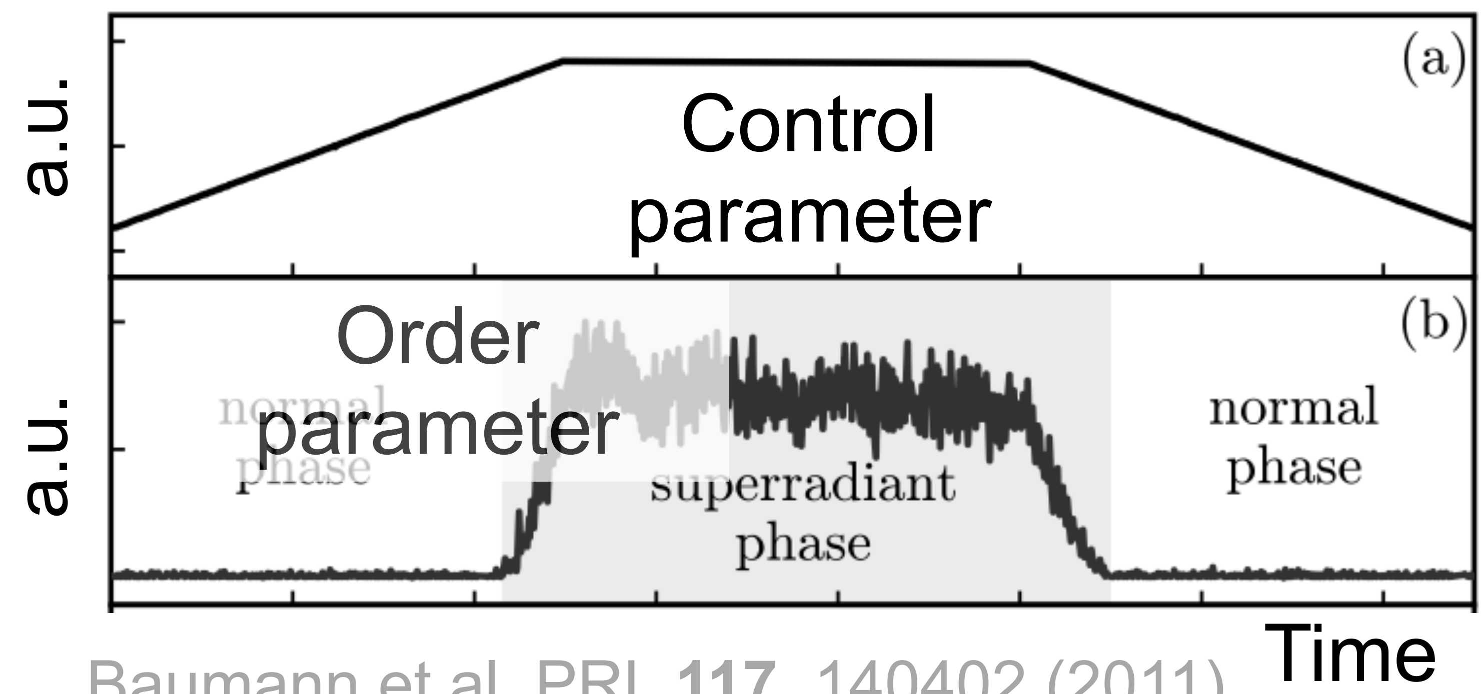
- No-go theorem for certain equilibrium transitions
  - Dicke super-radiant transition cannot be realised in equilibrium
  - Or even time crystals cannot be observed in equilibrium
  - Uncontrolled dissipative channels (e.g. spontaneous emissions)
- Reported non-equilibrium phase transition behaviour

<u>Numerics</u>



Kirton et al. PRL **118**, 123602 (2017)

<u>Experiments</u>



Baumann et al. PRL **117**, 140402 (2011)

Collective open quantum systems

- Quantum master equation: Heisenberg and functional dynamics

$$\dot{O}_t = \mathbb{L}^*[O_t] := i[H_N, O_t] + \sum_\mu \gamma_\mu \Big(J_\mu^\dagger\, O_t\, J_\mu - \frac{1}{2}\{O_t\,, J_\mu^\dagger J_\mu\} \Big)$$

Evolution of functionals

$$O_t = e^{t\mathbb{L}^*}[O] \qquad \mapsto \qquad \omega_t := \omega \circ e^{t\mathbb{L}^*}$$

$$\langle O \rangle_t = \omega(e^{t\mathbb{L}^*}[O]) = \omega_t(O)$$

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- Collective jump operators

$$J_\mu = \sqrt{\frac{1}{N}} \sum_{k=1}^{N} j^{(k)}$$

- Generic form of collective dynamical generators

$$V_\mu = \sum_{k=1}^{N} v_\mu^{(k)}$$

$$\mathbb{L}^*[O] = i[H_N, O] + \frac{1}{N} \sum_{\mu,\nu} c_{\mu\nu} \left(V_\mu\, O\, V_\nu - \frac{1}{2}\{O\,, V_\mu V_\nu\} \right)$$

$$\mathrm{Tr}(v_\mu\, v_\nu) = \delta_{\mu\nu}$$

$$H_N = \sum_\mu \Omega_\mu V_\mu + \sum_{\mu,\nu} \frac{h_{\mu\nu}}{N} V_\mu V_\nu$$

Example

$$v_\mu = \sigma_\mu / \sqrt{2}$$

Collective open quantum systems

- Generic form of collective dynamical generators

$$V_\mu = \sum_{k=1}^{N} v_\mu^{(k)}$$

$$\mathbb{L}^*[O] = i[H_N, O] + \frac{1}{N}\sum_{\mu,\nu} c_{\mu\nu}\left(V_\mu\, O\, V_\nu - \frac{1}{2}\left\{O\,, V_\mu V_\nu\right\}\right)$$

Kossakowski matrix

$$c_{\mu\nu} = a_{\mu\nu} + i b_{\mu\nu}\,, \quad a_{\mu\nu}\,, b_{\mu\nu} \in \mathbb{R}$$
$$c \geq 0$$

$$\mathbb{L}^*[O] = i[H_N, O] + \frac{1}{N}\sum_{\mu,\nu} \frac{a_{\mu\nu}}{2}\left[\left[V_\mu, O\right], V_\nu\right] + \frac{i}{N}\sum_{\mu\nu} \frac{b_{\mu\nu}}{2}\left\{\left[V_\mu, O\right], V_\nu\right\}$$

- Single-site dissipative contributions

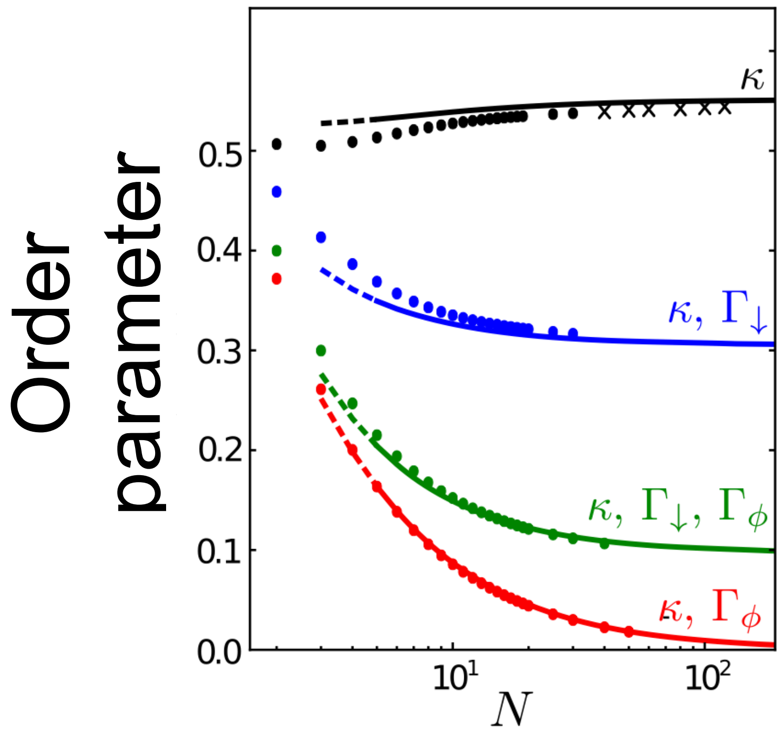
$$\mathbb{D}^*[O] = \gamma \sum_{k=1}^{N}\left(j^{\dagger\,(k)}\, O\, j^{(k)} - \frac{1}{2}\left\{j^{\dagger\,(k)}\, j^{(k)}, O\right\}\right)$$

Break the conservation of total angular momentum!

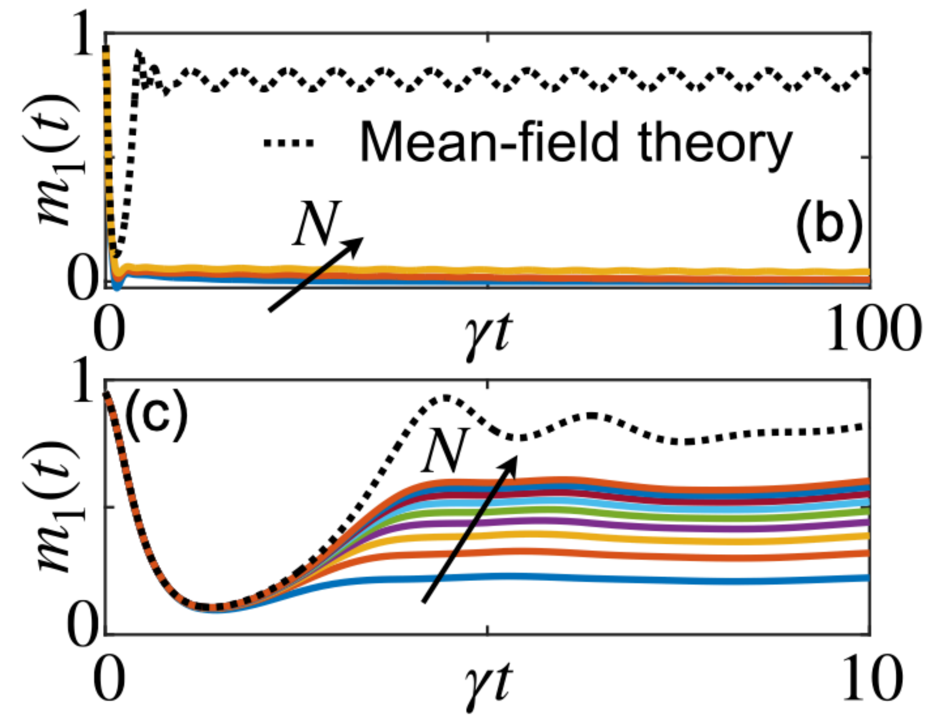
Collective open quantum systems

- Numerical simulation is still efficient but much more limited

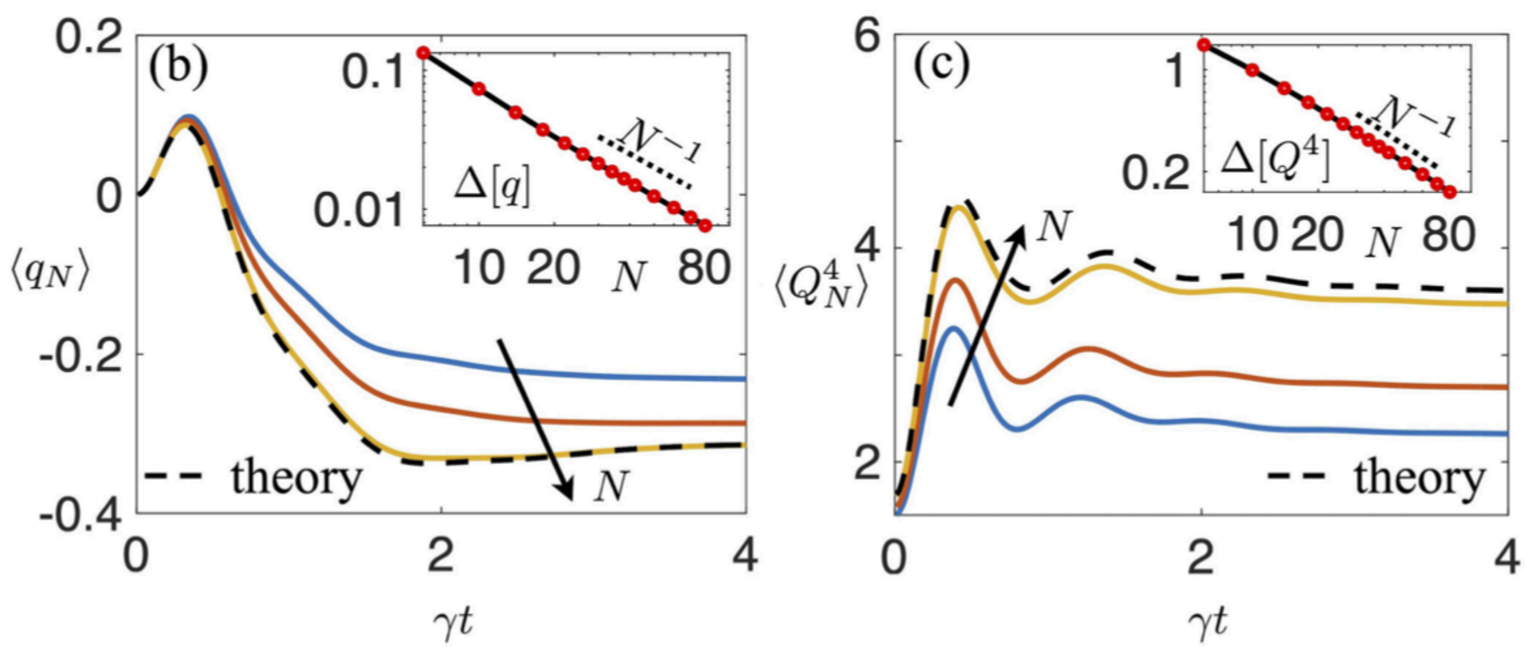
Chase et al. PRA **78**, 052101 (2008); Shammah et al. PRA **98**, 063815 (2018)



Kirton et al. PRL **118**, 123602 (2017)



FC et al. PRL **133**, 150401 (2024)



FC, PRL **131**, 227102 (2023)

Mean-field observables

- Sample-mean properties of the system

$$\omega\left(v_\mu^{(h)}\right) = \frac{1}{N}\sum_{k=1}^{N}\omega\left(v_\mu^{(k)}\right) = \omega\left(\frac{V_\mu}{N}\right)$$

$$\boxed{m_\mu^N := \frac{V_\mu}{N}}$$

$$m_\mu^N \to \ ?$$

$$N \gg 1$$

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$$\boxed{m_\mu^N := \frac{V_\mu}{N}}$$

$$m_\mu^N \to \,?$$
$$N \gg 1$$

- (Quantum) Law of large numbers
  - Clustering state ω

$$m_\mu^N \to \left[\lim_{N\to\infty}\omega(m_\mu^N)\right]\mathbf{1} = \omega(v_\mu)\mathbf{1}$$

    Under any possible expectation

$$\lim_{N\to\infty}\omega\left(A m_\mu^N B\right) = \omega(AB)\,\omega(v_\mu) \quad A, B \in \mathscr{A}_\infty$$

  - Proof

$$\left|\omega(A m_\mu^N B) - \omega(AB\, m_\mu^N)\right| \leq \|A\|\left\|\left[m_\mu^N, B\right]\right\| \to 0$$

$$\left|\omega\left(AB\left[m_\mu^N - \omega(v_\mu)\right]\right)\right| \leq \|A\|\,\|B\|\sqrt{\omega\left(\left[m_\mu^N - \omega(v_\mu)\right]^2\right)} \to 0$$

Mean-field observables

- Example: clustering state $\omega(O) = \langle \uparrow_{\text{all}} | O | \uparrow_{\text{all}} \rangle$

$$\omega\left(\left[m_\mu^N - \omega(v_\mu)\right]^2\right) = \frac{1}{N^2}\sum_{k,h=1}^{N}\left[\omega\left(v_\mu^{(k)} v_\mu^{(h)}\right) - \omega(v_\mu^{(k)})\omega(v_\mu^{(h)})\right]$$

$$= \frac{1}{N^2}\sum_{k=1}^{N}\left[\omega\left(v_\mu^{(k)} v_\mu^{(k)}\right) - \omega(v_\mu^{(k)})\omega(v_\mu^{(k)})\right]$$

$$= \frac{1}{N}\left[\omega\left(v_\mu^2\right) - \omega(v_\mu)^2\right] \to 0$$

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$$= \frac{1}{N}\left[\omega\left(v_\mu^2\right) - \omega(v_\mu)^2\right] \to 0$$

- Example: non-clustering state $\omega\left(O\right) = \langle S, m | O | S, m\rangle$

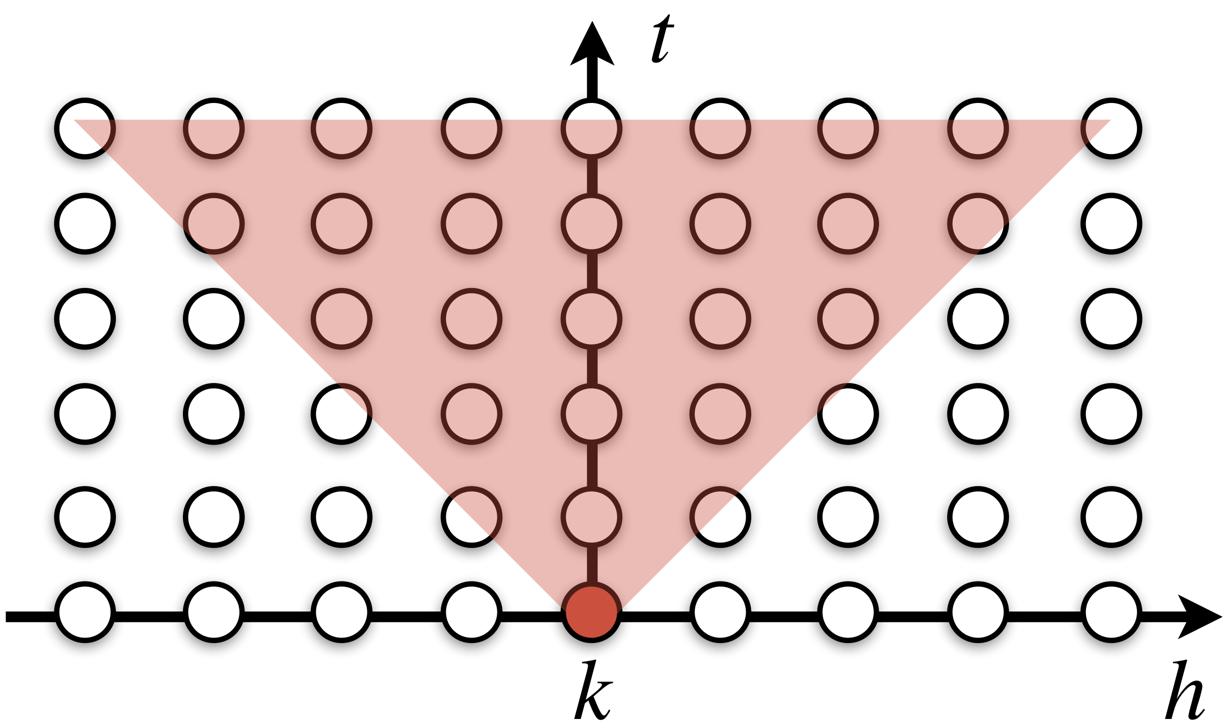
$$\omega\left(\left[m_{\mathrm{x}}^N - \omega(v_{\mathrm{x}})\right]^2\right) = \frac{1}{2N^2}\omega\left(S_{\mathrm{x}}^2\right) = \frac{1}{4N^2}\omega(S_{\mathrm{x}}^2 + S_{\mathrm{y}}^2) = \frac{1}{4N^2}\omega(S^2 - S_{\mathrm{z}}^2)$$

$$= \frac{1}{4N^2}\left[N(N+1) - m^2\right]$$

$$m = 0 \quad \rightarrow \quad \omega\left(\left[m_{\mathrm{x}}^N - \omega(v_{\mathrm{x}})\right]^2\right) \rightarrow \frac{1}{4}$$

Mean-field observables

- Dynamics with short-range interactions (Lieb-Robinson bounds)
  Lieb et al. CMP **28**, 251 (1972); Sweke et al. JPA 52, 424003 (2019)
  Poulin, PRL **104**, 190401 (2010)



Quasi-local evolution $\left\| \left[o^{(k)}(t), x^{(h)} \right] \right\| \approx e^{-\frac{|k-h|}{v\,t}}$

$\omega \circ e^{t\mathbb{L}^*}$ remains clustering

Law of large numbers $m_\mu^N(t) \to \left[\lim_{N\to\infty} \omega_t(m_\mu^N) \right] \mathbf{1}$

- The challenge is in determining their time-dependent value

$$\mathbb{L}^*[m_\beta^N] \propto \frac{i}{N} \sum_{k,h=1}^{N} \left[x^{(h)} x^{(h+1)}, v_\beta^{(k)} \right] = \frac{i}{N} \sum_{h=1}^{N} x^{(h)} [x, v_\beta]^{(h+1)} + \frac{i}{N} \sum_{h=1}^{N} x^{(h)} [x, v_\beta]^{(h-1)}$$

Collective open quantum systems

- Simplification for collective models

$$[v_\mu, v_\nu] = i \sum_\eta \varepsilon^\eta_{\mu\nu} v_\eta$$

$$\mathbb{L}^*[m^N_\beta] = -\sum_{\mu,\eta} \Omega_\mu \varepsilon^\eta_{\mu\beta} m^N_\eta - \sum_{\mu,\nu,\eta} h_{\mu\nu} \left(\varepsilon^\eta_{\nu\beta} m^N_\mu m^N_\eta + \varepsilon^\eta_{\mu\beta} m^N_\eta m^N_\nu \right)$$
$$- \sum_{\mu,\nu,\eta,\xi} \frac{a_{\mu\nu}}{2N} \varepsilon^\eta_{\mu\beta} \varepsilon^\xi_{\eta\nu} m^N_\xi - \sum_{\mu,\nu,\eta} \frac{b_{\mu\nu}}{2} \varepsilon^\eta_{\mu\beta} \left(m^N_\eta m^N_\nu + m^N_\nu m^N_\eta \right)$$

- If quantum state remains clustering $\omega_t\left(m^N_\mu m^N_\eta\right) \to \omega_t\left(m^N_\mu\right)\omega_t\left(m^N_\eta\right)$

Collective open quantum systems

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$$-\sum_{\mu,\nu,\eta,\xi} \frac{a_{\mu\nu}}{2N} \varepsilon^\eta_{\mu\beta} \varepsilon^\xi_{\eta\nu} m^N_\xi - \sum_{\mu,\nu,\eta} \frac{b_{\mu\nu}}{2} \varepsilon^\eta_{\mu\beta}\left(m^N_\eta m^N_\nu + m^N_\nu m^N_\eta\right)$$

- If quantum state remains clustering $\omega_t\left(m^N_\mu m^N_\eta\right) \to \omega_t\left(m^N_\mu\right)\omega_t\left(m^N_\eta\right)$

- Consider $m_\beta(t)$ that obeys the so-called mean-field equations of motion

$$\dot{m}_\beta(t) = -\sum_{\mu,\eta} \Omega_\mu \varepsilon^\eta_{\mu\beta} m_\eta(t) - \sum_{\mu,\nu,\eta} h_{\mu\nu}\left(\varepsilon^\eta_{\nu\beta} m_\mu(t) m_\eta(t) + \varepsilon^\eta_{\mu\beta} m_\eta(t) m_\nu(t)\right)$$
$$-\sum_{\mu,\nu,\eta} \frac{b_{\mu\nu}}{2} \varepsilon^\eta_{\mu\beta}\left(m_\eta(t) m_\nu(t) + m_\nu(t) m_\eta(t)\right)$$

Is this an approximation or is this exact?

Spohn, RMP **52**, 569 (1980); Alicki et al. J. Stat. Phys. **32**, 299 (1983)

Exactness of mean-field equations

- What do we have to prove?

$$\lim_{N\to\infty} \omega_t(m_\beta^N) = m_\beta(t)$$

$$\left| \omega_t(m_\beta^N) - m_\beta(t) \right| \leq \sqrt{\omega_t\left([m_\beta^N - m_\beta(t)]^2\right)}$$

FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

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- "Cost function"

$$\mathscr{E}_N(t) := \sum_\beta \omega_t\left([m_\beta^N - m_\beta(t)]^2\right)$$

controlling the limit to be proven

$$\left|\omega_t(m_\beta^N) - m_\beta(t)\right| \le \sqrt{\omega_t\left([m_\beta^N - m_\beta(t)]^2\right)} \le \sqrt{\mathscr{E}_N(t)}$$

$$\lim_{N\to\infty} \mathscr{E}_N(t) = 0 \quad \Longrightarrow \quad \lim_{N\to\infty} \omega_t(m_\beta^N) = m_\beta(t)$$

FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

Exactness of mean-field equations

- Fact:

  For the previously introduced collective models

- If $\lim_{N\to\infty} \mathscr{E}_N(0) = 0 \quad \Longrightarrow \quad \boxed{\lim_{N\to\infty} \mathscr{E}_N(t) = 0}$

FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

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  For the previously introduced collective models

  - If $\lim_{N\to\infty} \mathcal{E}_N(0) = 0 \quad \Longrightarrow \quad \boxed{\lim_{N\to\infty} \mathcal{E}_N(t) = 0}$

- Steps for the proof:

  - Bound to error growth $\dot{\mathcal{E}}_N(t) \le C_1 \mathcal{E}_N(t) + \dfrac{C_2}{N}$
  - Gronwall Lemma $\mathcal{E}_N(t) \le e^{tC_1}\mathcal{E}_N(0) + \dfrac{C_2}{C_1 N}\left(e^{C_1 t} - 1\right)$
  - Use the assumption $\lim_{N\to\infty} \mathcal{E}_N(t) \le e^{tC_1} \lim_{N\to\infty} \mathcal{E}_N(0) + \lim_{N\to\infty} \dfrac{C_2}{C_1 N}\left(e^{C_1 t} - 1\right)$

FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

Exactness of mean-field equations

- Derivation of the bound

$$\mathscr{E}_N(t) := \sum_\beta \omega_t\left([m_\beta^N - m_\beta(t)]^2\right)$$

$$\dot{\mathscr{E}}_N(t) = \sum_\beta \omega_t\left(\mathbb{L}^*\left[[m_\beta^N - m_\beta(t)]^2\right]\right) - 2\sum_\beta \dot{m}_\beta(t)\,\omega_t\left(m_\beta^N - m_\beta(t)\right)$$

- Use that $\mathbb{L}^*[XY] = \mathbb{L}^*[X]Y + X\mathbb{L}^*[Y] + \sum_{\mu,\nu}\frac{c_{\mu\nu}}{N}[V_\mu, X][Y, V_\nu]$

$$\dot{\mathscr{E}}_N(t) = \sum_\beta \omega_t\left(\left(\mathbb{L}^*[m_\beta^N] - \dot{m}_\beta(t)\right)\left(m_\beta^N - m_\beta(t)\right)\right) + \mathrm{c.c.} + O\left(\frac{1}{N}\right)$$

FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

Exactness of mean-field equations

- Derivation of the bound

$$\mathbb{L}^*[m_\beta^N] - \dot{m}_\beta(t) = \sum_s q_s\left(m_{\eta_s}^N - m_{\eta_s}(t)\right) + \sum_s p_s\left(m_{\eta_s}^N m_{\mu_s}^N - m_{\eta_s}(t)\, m_{\mu_s}(t)\right) + O\left(\frac{1}{N}\right)$$

$$\left(m_{\eta_s}^N - m_{\eta_s}(t)\right) m_{\mu_s}^N + \left(m_{\mu_s}^N - m_{\mu_s}(t)\right) m_{\eta_s}(t)$$

- Putting things together

$$\dot{\mathscr{E}}_N(t) = \sum_s r_s\, \omega_t\left(\left(m_{\eta_s}^N - m_{\eta_s}(t)\right) X_s \left(m_{\mu_s}^N - m_{\mu_s}(t)\right)\right) + O\left(\frac{1}{N}\right)$$

$$\left|\omega_t\left(\left(m_{\eta_s}^N - m_{\eta_s}(t)\right) X_s \left(m_{\mu_s}^N - m_{\mu_s}(t)\right)\right)\right| \leq \|X_s\| \underbrace{\sqrt{\omega_t\left(\left(m_{\eta_s}^N - m_{\eta_s}(t)\right)^2\right)}}_{\leq \sqrt{\mathscr{E}_N(t)}} \underbrace{\sqrt{\omega_t\left(\left(m_{\mu_s}^N - m_{\mu_s}(t)\right)^2\right)}}_{\leq \sqrt{\mathscr{E}_N(t)}}$$

FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

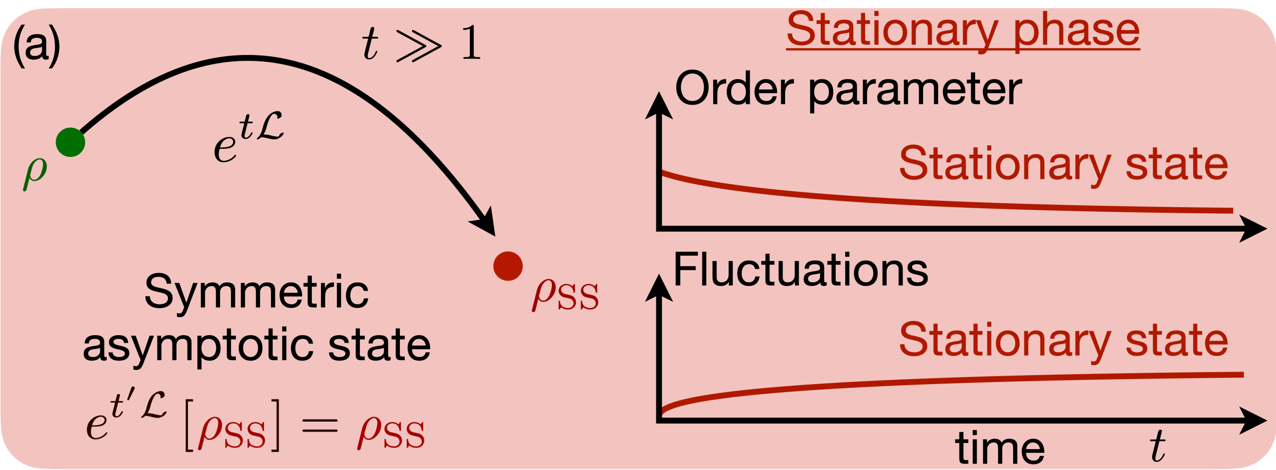
Application: Time crystals

- Dissipative (continuous) time crystals

  Wilczek, PRL **109**, 160401 (2012)

https://www.youtube.com/watch?v=TWhWYAnKknw

$$\rho(t) = \mathbb{L}[\rho(t)]$$



- Time-translation symmetric phase

$$\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} e^{t\mathbb{L}}[\rho(0)] = \rho_{SS}$$

Time-translation is a symmetry

$$e^{t\mathbb{L}} \circ \mathbb{L} = \mathbb{L} \circ e^{t\mathbb{L}}$$

Symmetric stationary state

$$e^{t\mathbb{L}}[\rho_{SS}] = \rho_{SS}$$

FC et al. PRA **105**, L040202 (2022)

Analogy with Hamiltonian systems

$\mathbb{L} \leftrightarrow H$ Hamiltonian

Symmetry of the Hamiltonian

$e^{t\mathbb{L}} \leftrightarrow U$ $[U, H] = 0$

Symmetric ground state

$\rho_{SS} \leftrightarrow |\psi_{GS}\rangle$

Application: Time crystals

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Wilczek, PRL **109**, 160401 (2012)

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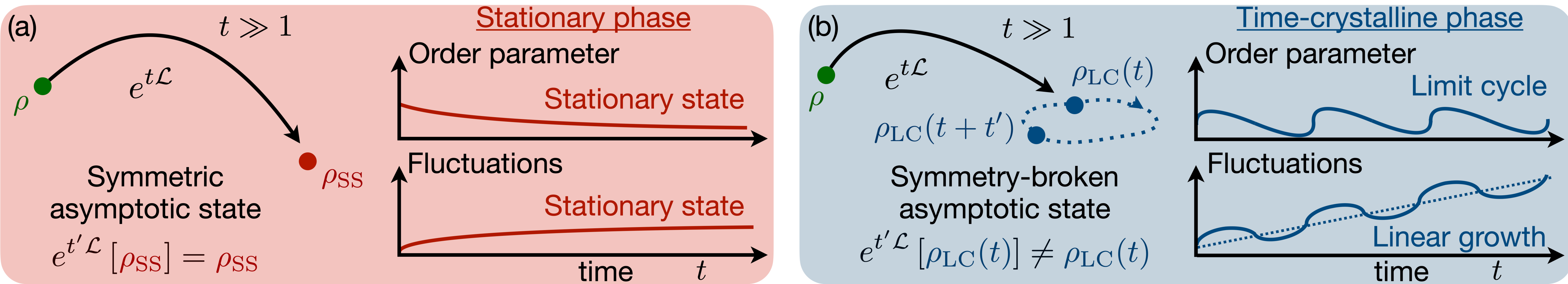
$$\rho(t) = \mathbb{L}[\rho(t)]$$



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Time-translation is a symmetry

$$e^{t\mathbb{L}} \circ \mathbb{L} = \mathbb{L} \circ e^{t\mathbb{L}}$$

Symmetric stationary state

$$e^{t\mathbb{L}}[\rho_{\mathrm{SS}}] = \rho_{\mathrm{SS}}$$

- Time-translation symmetry breaking

$$\nexists \lim_{t\to\infty} \rho(t) :\ \rho(t) \to \rho_{\mathrm{LC}}(t)$$

Time-translation is a symmetry

$$e^{t\mathbb{L}} \circ \mathbb{L} = \mathbb{L} \circ e^{t\mathbb{L}}$$

Stationary state breaks the symmetry

$$e^{t'\mathbb{L}}[\rho_{\mathrm{LC}}(t)] \neq \rho_{\mathrm{LC}}(t)$$

FC et al. PRA **105**, L040202 (2022)

Application: Time crystals

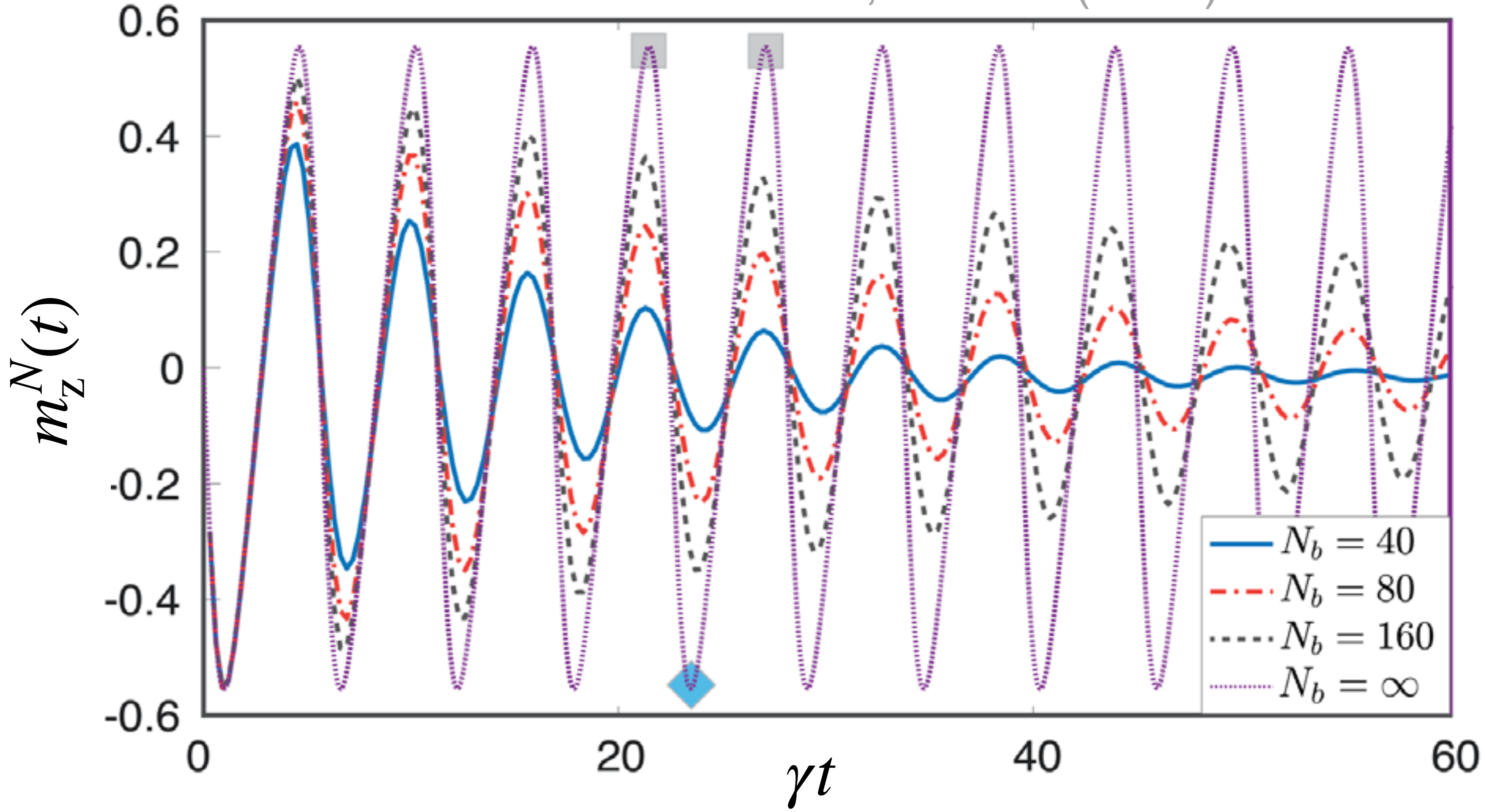
- Boundary time crystal

$$H_N = \frac{\Omega}{2}\sum_{k=1}^{N}\sigma_{\mathrm{x}}^{(k)} \qquad J = \sqrt{\frac{\gamma}{2N}}\sum_{k=1}^{N}\sigma_{-}^{(k)}$$

Ferioli et al. Nat. Phys. **19**, 1345 (2023)

- Numerics

Iemini et al. PRL **121**, 035301 (2018)



- Is this really a time-crystal phase transition?

Mean-field equations are exact in the thermodynamic limit!

Application: Time crystals

- Boundary time crystal
  - Mean-field equations

$$\dot{m}_1(t) = \sqrt{2}\gamma m_1(t) m_3(t)$$
$$\dot{m}_2(t) = \sqrt{2}\gamma m_2(t) m_3(t) - \Omega m_3(t)$$
$$\dot{m}_3(t) = \Omega m_2(t) - \sqrt{2}\gamma [m_1^2(t) + m_2^2(t)]$$

Note that equations can be derived from an effective Hamiltonian

FC et al. PRA **105**, L040202 (2022)

Application: Time crystals

- Boundary time crystal
  - Mean-field equations

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Note that equations can be derived from an effective Hamiltonian

  - Assume $m_1(0) = 0 \mapsto m_1(t) = 0$

$$m_2(t) = \cos[f(t)]\, m_2(0) + \sin[f(t)]\, m_3(0)$$
$$m_3(t) = \cos[f(t)]\, m_3(0) - \sin[f(t)]\, m_2(0)$$

FC et al. PRA **105**, L040202 (2022)

Application: Time crystals

- Boundary time crystal
  - Mean-field equations

$$\dot{m}_1(t) = \sqrt{2}\gamma m_1(t) m_3(t)$$
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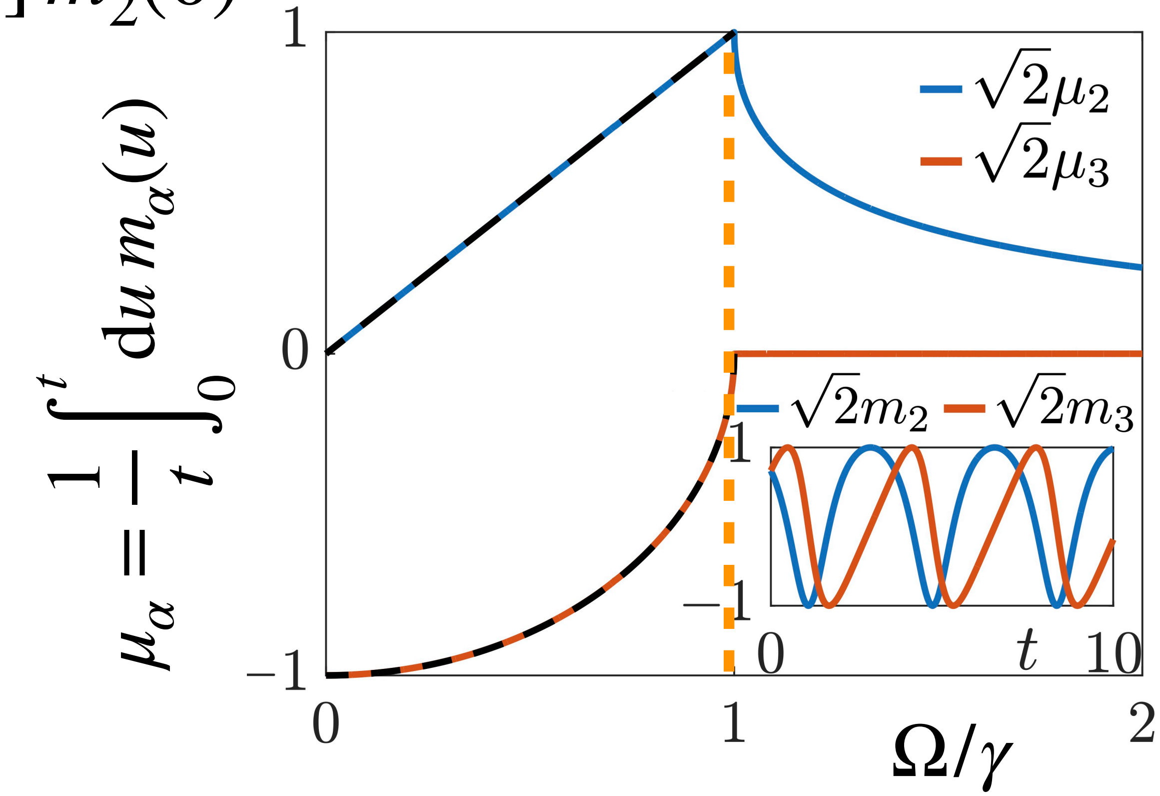
  - Two regimes

Stationary phase $\lim_{t\to\infty} f(t) = f(\infty)$

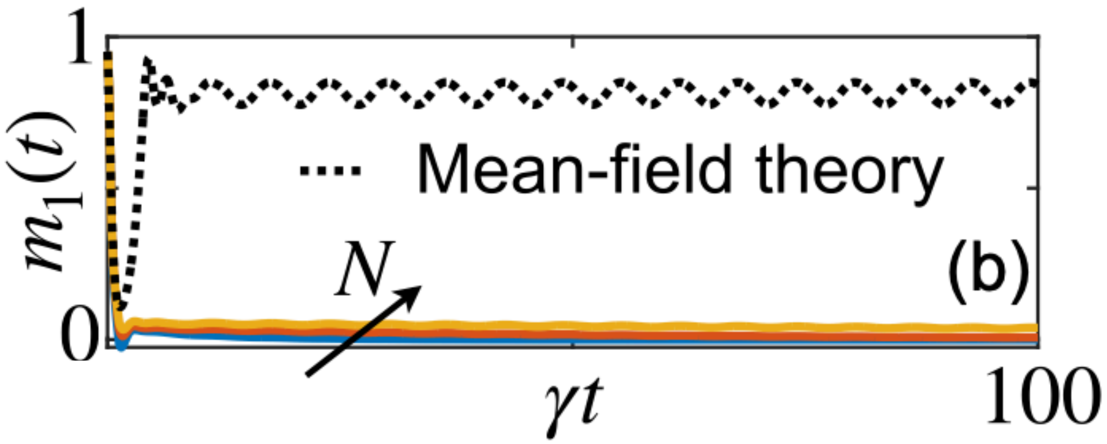
Time crystal $\nexists \lim_{t\to\infty} f(t)$



FC et al. PRA **105**, L040202 (2022)

Remark

- Subtlety: time vs system size limit



FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

Remark

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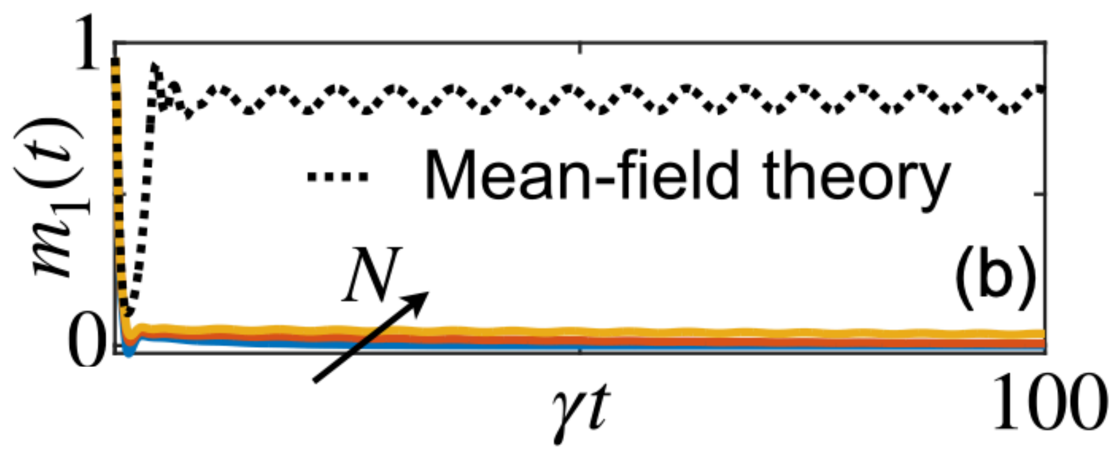
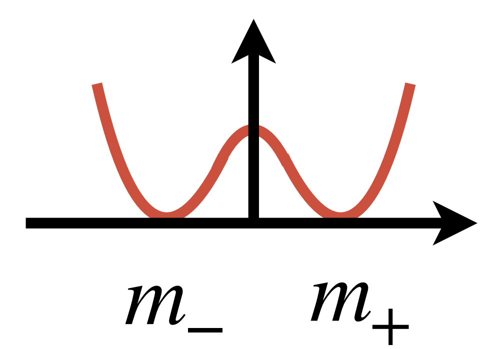


- Simple mean-field equation

$$\mathbb{L}^* \quad \mapsto \quad \dot{m}(t) = -m(t) + \tanh[\beta m(t)]$$

Single stationary state for any finite system

Below critical temperature



FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

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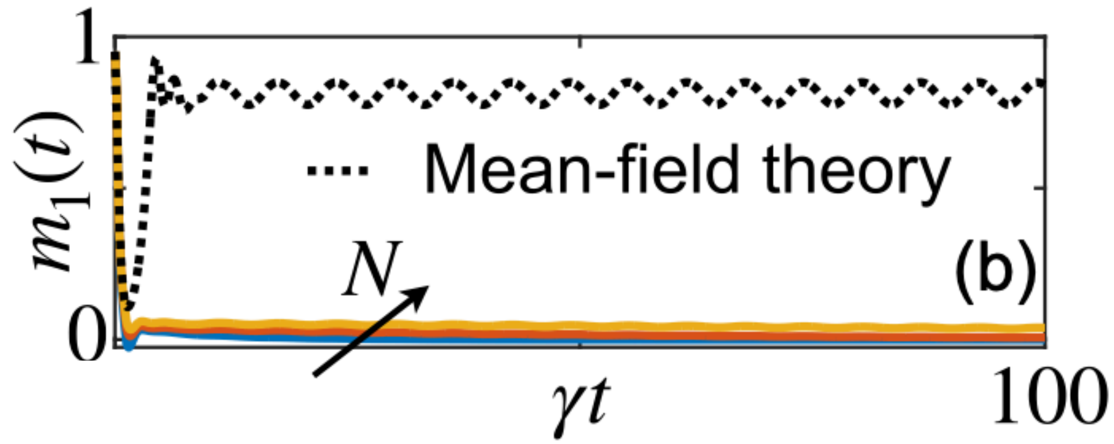

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$\downarrow$

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$\downarrow$

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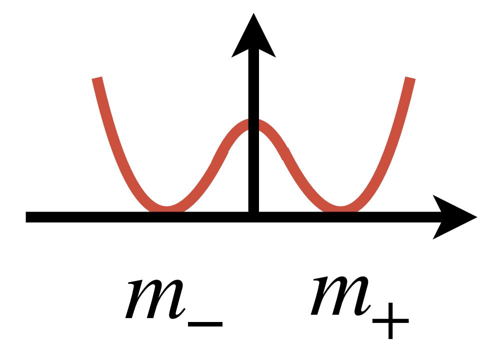
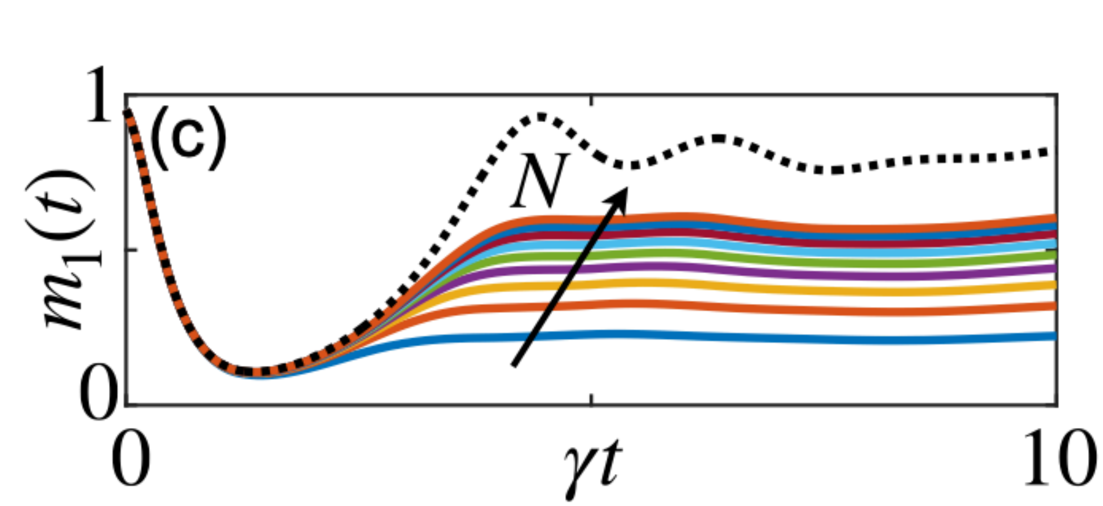


- Correct interpretation $\mathscr{E}_N(t) \sim \dfrac{e^{tC_1}}{N}$

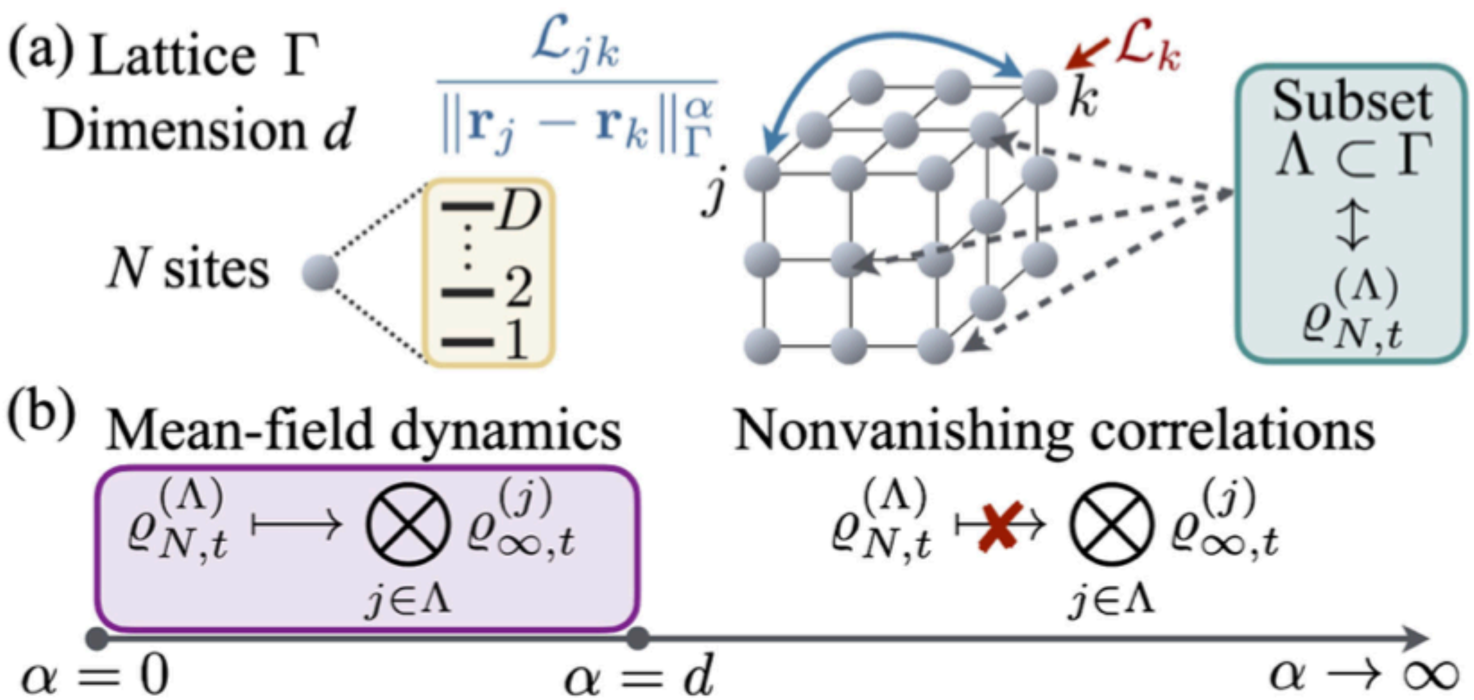
For any finite time, there is N* large enough …



FC et al. PRL **126**, 230601 (2021) & PRL **133**, 150401 (2024)

Long-range interactions

- What about long-range interactions but not collective?



- Heisenberg equations are not functions of mean-field observables

$$\mathbb{L}^*[m_\beta^N] \neq f_\beta\left(\left\{m_\mu^N\right\}_\mu\right)$$

- Different approach: Can the dynamics generate local correlations?

$$\varrho_N^\Lambda(t) \quad \mapsto \quad \bigotimes_{k\in\Lambda} \varrho_\infty^{(k)}(t) \quad ?$$

Mattes et al. PRL **134**, 070402 (2025)

Long-range interactions

- Evolution equation for the reduced density matrix

$$\dot{\varrho}_N^{\Lambda}(t) = \sum_{j\in\Lambda} \mathbb{L}_j^{(j)}[\varrho_N^{\Lambda}(t)] + \frac{1}{c_\alpha^N} \sum_{j\in\Lambda, k\notin\Lambda} \frac{1}{\|\mathbf{r}_k - \mathbf{r}_j\|^\alpha} \mathrm{Tr}_k \left(i[x^{(j)}x^{(k)}, \varrho_N^{\Lambda\cup\{k\}}(t)] \right)$$

$$+ \frac{1}{2c_\alpha^N} \sum_{j\neq k\in\Lambda} \frac{1}{\|\mathbf{r}_k - \mathbf{r}_j\|^\alpha} \mathrm{Tr}_k \left(i[x^{(j)}x^{(k)}, \varrho_N^{\Lambda\cup\{k\}}(t)] \right)$$

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$$\cancel{+\frac{1}{2c_\alpha^N} \sum_{j\neq k\in\Lambda} \frac{1}{\|\mathbf{r}_k - \mathbf{r}_j\|^\alpha} \mathrm{Tr}_k \left(i[x^{(j)}x^{(k)}, \varrho_N^{\Lambda\cup\{k\}}(t)] \right)} \qquad \begin{array}{c} \alpha \leq d \\ \propto 1/c_\alpha^N \end{array}$$

- Hierarchy of equations $\alpha \leq d$

$$\dot{\varrho}_N^{\Lambda}(t) \approx \sum_{j\in\Lambda} \mathbb{L}_j^{(j)}[\varrho_N^{\Lambda}(t)] + \frac{1}{c_\alpha^N} \sum_{j\in\Lambda, k\notin\Lambda} \frac{1}{\|\mathbf{r}_k - \mathbf{r}_j\|^\alpha} \mathrm{Tr}_k \left(i[x^{(j)}x^{(k)}, \varrho_N^{\Lambda\cup\{k\}}(t)] \right)$$

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$$\dot{\varrho}_N^{\Lambda}(t) \approx \sum_{j\in\Lambda} \mathbb{L}_j^{(j)}[\varrho_N^{\Lambda}(t)] + \frac{1}{c_\alpha^N} \sum_{j\in\Lambda, k\notin\Lambda} \frac{1}{\|\mathbf{r}_k - \mathbf{r}_j\|^\alpha} \mathrm{Tr}_k \left(i[x^{(j)}x^{(k)}, \varrho_N^{\Lambda\cup\{k\}}(t)] \right)$$

  - Solved by a family of factorised states $\alpha \leq d$

$$\varrho_\infty^{\Lambda}(t) = \bigotimes_{k\in\Lambda} \varrho_\infty^{(k)}(t)$$

Example $\quad \dot{\varrho}_\infty^{(1)}(t) = \mathbb{L}_1[\varrho_\infty^{(1)}(t)] + \mathrm{Tr}\left(\mathbb{L}_{12}[\varrho_\infty^{(1)}(t) \otimes \varrho_\infty^{(1)}(t)] \right)$

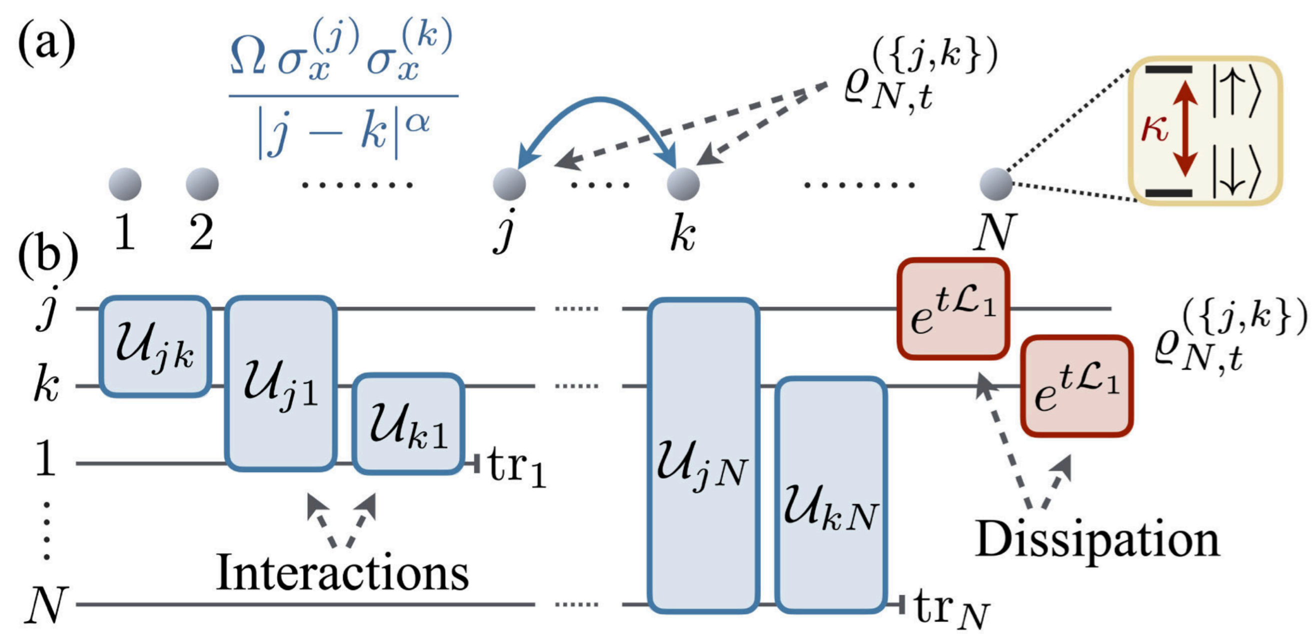
Mattes et al. PRL **134**, 070402 (2025)

Long-range interactions

- Numerical benchmark
  - System

$$H_N = \frac{\Omega}{c_\alpha^N} \sum_{j,k=1}^{N} \frac{\sigma_x^{(j)}\sigma_x^{(k)}}{|j-k|^\alpha}, \quad \mathbb{D}[O] = \kappa \sum_{k=1}^{N} \left(\sigma_x^{(k)} O \sigma_x^{(k)} - O \right)$$

  - Circuit representation



Mattes et al. PRL **134**, 070402 (2025)

Long-range interactions

- Numerical benchmark
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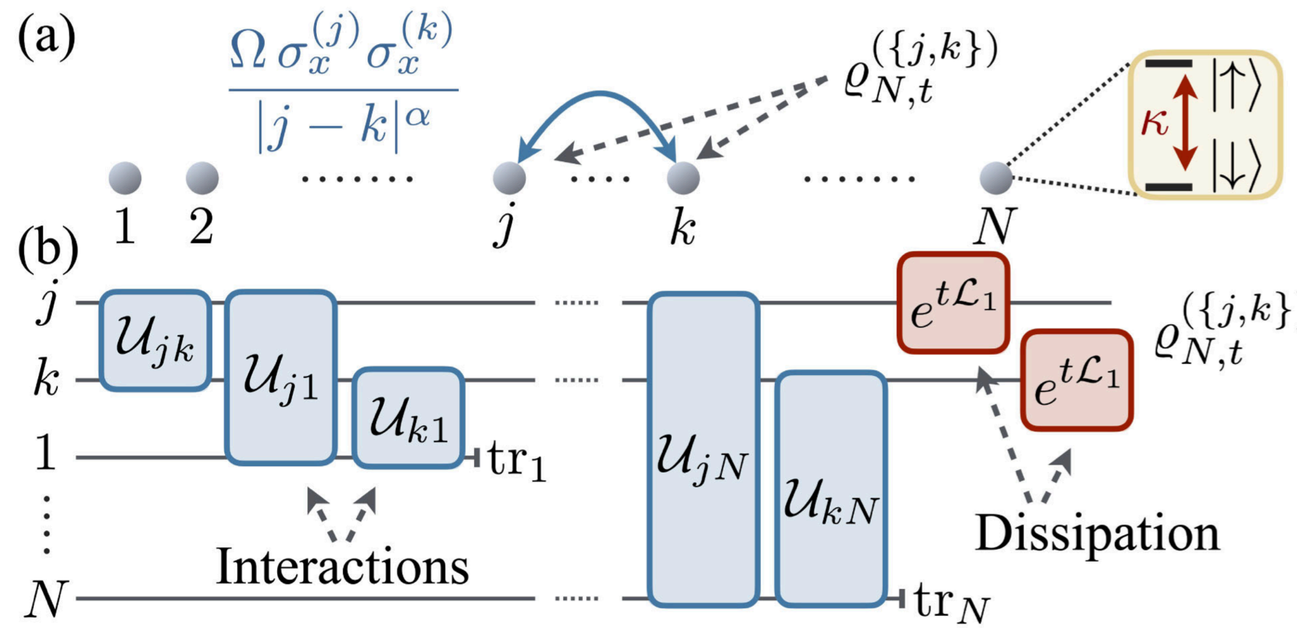
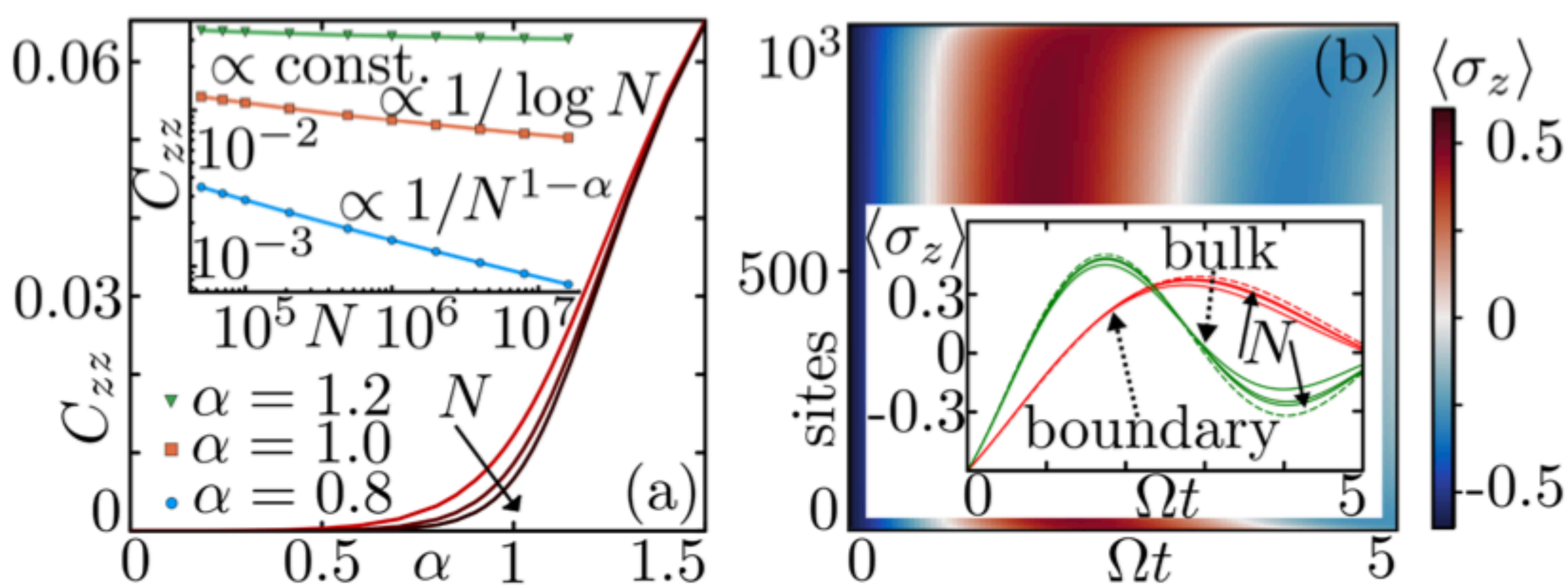


  - Results



Mattes et al. PRL **134**, 070402 (2025)

Intermezzo

Quantum fluctuations

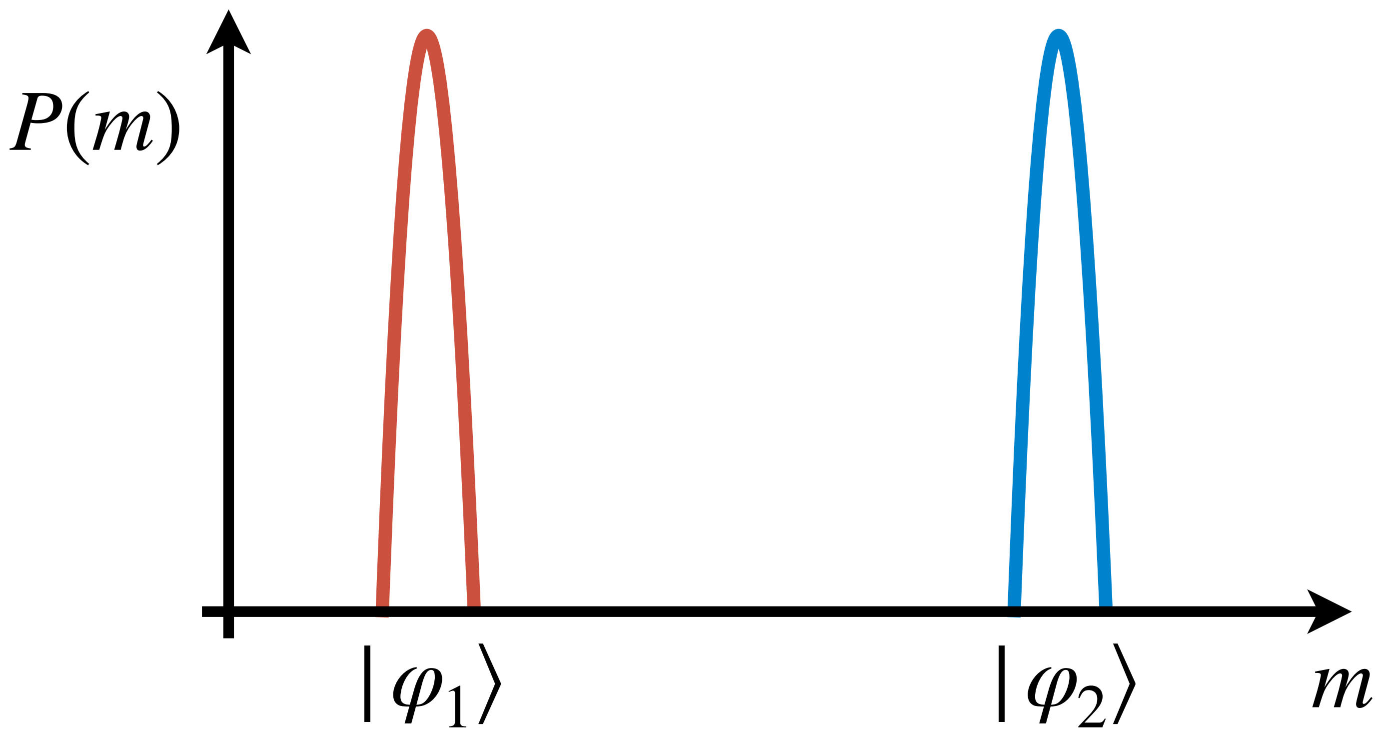
- What is the problem with mean-field (sample-mean) observables?
  - Suitable order parameters ✔
  - Classical variables, no correlations ✘

$$\left[m_\mu^N, m_\nu^N\right] = \frac{i}{N^2}\sum_{k,h=1}^{N}\left[v_\mu^{(k)}, v_\nu^{(h)}\right] = \frac{i}{N}\sum_\eta \varepsilon_{\mu\nu}^\eta\, m_\eta^N \to 0$$

- How can we consider quantum/classical correlations within a “phase”?

Superposition state

$$|\psi\rangle \propto |\varphi_1\rangle + |\varphi_2\rangle$$



Quantum fluctuations

- From law of large numbers to (quantum) central limit theorem

$$m_\mu^N = \frac{1}{N}\sum_{k=1}^{N} v_\mu^{(k)} \qquad \rightarrow \qquad F_\mu^N = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\left[v_\mu^{(k)} - \omega\left(v_\mu^{(k)}\right)\right]$$

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

- From law of large numbers to (quantum) central limit theorem

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- Susceptibility (scaled cumulant of order-parameter)

$$\omega\left(F_\mu^N F_\mu^N\right) = \frac{1}{N}\sum_{k,h=1}^{N}\left[\omega\left(v_\mu^{(k)} v_\mu^{(h)}\right) - \omega\left(v_\mu^{(k)}\right)\omega\left(v_\mu^{(h)}\right)\right]$$

$$\omega\left(F_\mu^N F_\mu^N\right) = N\,\omega\left(\left[m_\mu^N - \omega(m_\mu^N)\right]^2\right) =: \chi_\mu$$

- Collective correlations (not captured locally)

$$\omega\left(v_\mu^{(k)} v_\mu^{(h)}\right) - \omega\left(v_\mu^{(k)}\right)\omega\left(v_\mu^{(h)}\right) \propto \frac{1}{N}$$

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

- Preserved quantum character

$$\left[F_\mu^N, F_\nu^N\right] = \frac{1}{N}\sum_{k,h=1}^{N}\left[v_\mu^{(k)}, v_\nu^{(h)}\right] = i\sum_\eta \varepsilon_{\mu\nu}^\eta\, m_\eta^N \quad \rightarrow \quad \left[i\sum_\eta \varepsilon_{\mu\nu}^\eta\, \omega(v_\eta)\right]\mathbf{1}$$

For clustering states

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

- Preserved quantum character

$$\left[F_\mu^N, F_\nu^N\right] = \frac{1}{N}\sum_{k,h=1}^{N}\left[v_\mu^{(k)}, v_\nu^{(h)}\right] = i\sum_\eta \varepsilon_{\mu\nu}^\eta m_\eta^N \quad \rightarrow \quad \left[i\sum_\eta \varepsilon_{\mu\nu}^\eta \omega(v_\eta)\right]\mathbf{1}$$

For clustering states

- Emergent bosonic description of the spin system

$$F_\mu^N \quad \rightarrow \quad B_\mu$$

Quadrature operators

- Compare with Holstein-Primakoff approximation

$$S_+ = \sqrt{N}\sqrt{1-\frac{a^\dagger a}{N}}\,a \approx \sqrt{N}a$$

Around the pure state all up!

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

- Example $$\omega\left(O\right) = \langle \uparrow_{\text{all}} | O | \uparrow_{\text{all}} \rangle$$

$$\omega\left(\sigma_{\text{x/y}}^{(k)}\right) = 0 \qquad \omega\left(\sigma_{\text{z}}^{(k)}\right) = 1$$

  - Commutation relations

$$\left[F_{\text{x}}^{N}, F_{\text{y}}^{N}\right] = \frac{i}{N}\sum_{k=1}^{N}\sigma_{\text{z}}^{(k)} \quad \to \quad i$$

Reminiscent of $\left[x, p\right] = i$

  - Covariance

$$\omega\left(F_{\text{x}}^{N}F_{\text{x}}^{N}\right) \to \frac{1}{2}$$

$$\omega\left(F_{\text{y}}^{N}F_{\text{y}}^{N}\right) \to \frac{1}{2}$$

Reminiscent of the bosonic vacuum state $|0\rangle$

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

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$$\left[F^N_{\text{x}}, F^N_{\text{y}}\right] = \frac{i}{N}\sum_{k=1}^{N}\sigma^{(k)}_{\text{z}} \quad \to \quad i$$

Reminiscent of $[x,p] = i$

  - Covariance

$$\omega\left(F^N_{\text{x}} F^N_{\text{x}}\right) \to \frac{1}{2}$$

$$\omega\left(F^N_{\text{y}} F^N_{\text{y}}\right) \to \frac{1}{2}$$

Reminiscent of the bosonic vacuum state $|0\rangle$

- Central limit theorem (mapping)

$$\lim_{N\to\infty} \omega\left(e^{ir_1 F^N_{\text{x}} + ir_2 F^N_{\text{y}}}\right) = e^{-(r_1^2+r_2^2)/4} = \langle 0 | e^{ir_1 x + ir_2 p} | 0 \rangle$$

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

- Central limit theorem (derivation)

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  - Simplify the expression

$$\omega\left(e^{ir_1 F_x^N + ir_2 F_y^N}\right) = \omega\left(\prod_{k=1}^{N} \exp\left[\frac{i}{\sqrt{2N}}\left(r_1\sigma_x^{(k)} + r_2\sigma_y^{(k)}\right)\right]\right)$$

$$= \omega\left(\exp\left[\frac{i}{\sqrt{2N}}\left(r_1\sigma_x + r_2\sigma_y\right)\right]\right)^N$$

Quantum fluctuations

- Central limit theorem (derivation)

$$\lim_{N\to\infty} \omega\left(e^{ir_1 F_x^N + ir_2 F_y^N}\right) = e^{-(r_1^2+r_2^2)/4} = \langle 0 | e^{ir_1 x + ir_2 p} | 0 \rangle$$

  - Simplify the expression

$$\omega\left(e^{ir_1 F_x^N + ir_2 F_y^N}\right) = \omega\left(\prod_{k=1}^{N} \exp\left[\frac{i}{\sqrt{2N}}\left(r_1\sigma_x^{(k)} + r_2\sigma_y^{(k)}\right)\right]\right)$$

$$= \omega\left(\exp\left[\frac{i}{\sqrt{2N}}\left(r_1\sigma_x + r_2\sigma_y\right)\right]\right)^N$$

  - Note that

$$\omega\left(\exp\left[\frac{i}{\sqrt{2N}}\left(r_1\sigma_x + r_2\sigma_y\right)\right]\right) \approx \omega\left(\mathbf{1} + \frac{i}{\sqrt{2N}}(r_1\sigma_x + r_2\sigma_y) - \frac{1}{4N}(r_1^2 + r_2^2)\right)$$

  - Therefore

$$\omega\left(e^{ir_1 F_x^N + ir_2 F_y^N}\right) \approx \left(1 - \frac{r_1^2 + r_2^2}{4N}\right)^N$$

Quantum fluctuations

- Quantum central limit theorem (in general)
  - Quantum fluctuations $\{F_\mu^N\}_\mu$
  - Clustering* state ω
  - Symplectic matrix $$s_{\mu\nu} = -i \lim_{N\to\infty} \left(\omega \left[F_\mu^N, F_\nu^N \right] \right)$$
  - Covariance matrix $$\Sigma_{\mu\nu} = \lim_{N\to\infty} \frac{1}{2} \omega \left(\left\{ F_\mu^N, F_\nu^N \right\} \right)$$

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Quantum fluctuations

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  - Covariance matrix $$\Sigma_{\mu\nu} = \lim_{N\to\infty} \frac{1}{2} \omega \left(\left\{ F_\mu^N, F_\nu^N \right\} \right)$$
- Theorem

$$\lim_{N\to\infty} \omega \left(e^{i\left(r, F^N\right)} \right) = \exp\left(-\frac{(r, \Sigma r)}{2} \right) = \hat{\omega}\left(e^{i(r,B)} \right)$$

  - Where $[B_\mu, B_\nu] = i s_{\mu\nu}$

Goderis et al. CMP **128**, 533 (1990); Benatti et al. JPA **51**, 325001 (2018)

Dynamics of quantum fluctuations

- Dynamics of quantum fluctuations in collective systems
  - Define quantum fluctuations

$$F_\mu^N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left[v_\mu^{(k)} - \omega_t \left(v_\mu^{(k)} \right) \right]$$

  - Fact

$$\boxed{\lim_{N\to\infty} \omega_t \left(e^{i\left(r,\, F^N\right)} \right) = \exp\left(-\frac{\left(r, \Sigma(t)\, r\right)}{2} \right) = \hat{\omega}_t \left(e^{i(r,B)} \right)}$$

  - Time-evolved covariance matrix

$$\dot{\Sigma}(t) = G(t)\Sigma(t) + \Sigma(t)G^T(t) + s(t)as^T(t)$$

$$G(t) = D(t) + s(t)\left[2h + b\right]$$

Mean-field dynamics $\dot{m}(t) = D(t)m(t)$

Benatti et al. JPA **51**, 325001 (2018)

Dynamics of quantum fluctuations

- "Normal" modes vary in time
  - Time-dependent symplectic matrix

Example
$$s(t)=\begin{pmatrix}0 & m_{\mathrm{z}}(t) & -m_{\mathrm{y}}(t)\\ -m_{\mathrm{z}}(t) & 0 & m_{\mathrm{x}}(t)\\ m_{\mathrm{y}}(t) & -m_{\mathrm{x}}(t) & 0\end{pmatrix}$$

  - "Rotate" into canonical form

$$R(t)s(t)R^T(t) \quad \mapsto \quad \begin{pmatrix}0 & \lambda & 0\\ -\lambda & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

  - Rescale to standard commutation relations

$$F^N_{\mathrm{x,y}} \to \frac{1}{\sqrt{\lambda}}F^N_{\mathrm{x,y}} \quad \mapsto \quad \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

FC et al. PRA **105**, L040202 (2022)

Dynamics of quantum fluctuations

- Evolution of the normal modes (if initial state is already "normal")

  Frame rotating with the mean-field observables

$$\tilde{X} = R(t) X R^T(t)$$

$$\dot{\tilde{\Sigma}}(t) = \tilde{G}(t)\tilde{\Sigma}(t) + \tilde{\Sigma}(t)\tilde{G}^T(t) + s(0)\tilde{a}(t)s^T(0)$$

$$\tilde{G}(t) = s(0)\left[2\tilde{h}(t) + \tilde{b}(t)\right]$$

- Emergent bosonic dynamical generator

$$\mathbb{L}^*[O] = i\sum_{\mu,\nu}\tilde{h}_{\mu\nu}(t)\left[B_\mu B_\nu, O\right] + \sum_{\mu\nu}\tilde{c}(t)\left(B_\mu O B_\nu - \frac{1}{2}\left\{B_\mu B_\nu, O\right\}\right)$$

$$\tilde{c}(t) = \tilde{a}(t) + i\tilde{b}(t)$$

FC et al. PRA **105**, L040202 (2022)

Dynamics of quantum fluctuations

- Derivation of the dynamics of the covariance matrix

$$\Sigma^N_{\alpha\beta}(t) = \frac{1}{2}\omega_t\left(\left\{F^N_\alpha, F^N_\beta\right\}\right)$$

- Define the two-point function $C^N_{\alpha\beta}(t) = \omega_t\left(F^N_\alpha\, F^N_\beta\right)$

Buonaiuto et al. PRL **127**, 133601 (2021)

Dynamics of quantum fluctuations

- Derivation of the dynamics of the covariance matrix

$$\Sigma^N_{\alpha\beta}(t) = \frac{1}{2}\omega_t\left(\left\{F^N_\alpha, F^N_\beta\right\}\right)$$

- Define the two-point function $C^N_{\alpha\beta}(t) = \omega_t\left(F^N_\alpha F^N_\beta\right)$

- And study its evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}C^N_{\alpha\beta}(t) = \omega_t\left(\mathbb{L}^*\left[F^N_\alpha F^N_\beta\right]\right) + \omega_t\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}F^N_\alpha\right)F^N_\beta\right) + \omega_t\left(F^N_\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}F^N_\beta\right)\right)$$

- Note that $\frac{\mathrm{d}}{\mathrm{d}t}F^N_\alpha = -\frac{1}{\sqrt{N}}\frac{\mathrm{d}}{\mathrm{d}t}\omega_t\left(V_\alpha\right)$ and $\omega_t\left(F^N_\beta\right) = 0$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}C^N_{\alpha\beta}(t) = \omega_t\left(\mathbb{L}^*\left[F^N_\alpha F^N_\beta\right]\right)}$$

Buonaiuto et al. PRL **127**, 133601 (2021)

Dynamics of quantum fluctuations

- Drift terms
  - Terms of the type $\omega_t\left(\mathbb{H}^*\left[F^N_\alpha\right]\,F^N_\beta\right)$

$$\mathbb{H}^*\left[F^N_\alpha\right] \propto m^N_\eta \frac{1}{\sqrt{N}} V_\mu = \left[m^N_\eta - \omega_t(m^N_\eta)\right] \frac{1}{\sqrt{N}} V_\mu + \omega_t(m^N_\eta) \frac{1}{\sqrt{N}} V_\mu$$

$$= \left[m^N_\eta - \omega_t(m^N_\eta)\right] F^N_\mu + F^N_\eta \omega_t(m^N_\mu) + \omega_t(m^N_\eta) \frac{1}{\sqrt{N}} V_\mu$$

Buonaiuto et al. PRL **127**, 133601 (2021)

Dynamics of quantum fluctuations

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$$= \left[m^N_\eta - \omega_t(m^N_\eta)\right] F^N_\mu + F^N_\eta \omega_t(m^N_\mu) + \omega_t(m^N_\eta)\frac{1}{\sqrt{N}} V_\mu$$

  - Note that

$$\omega_t\left(\mathbb{H}^*\left[F^N_\alpha\right]\, F^N_\beta\right) = \omega_t\left(\mathbb{H}^*\left[F^N_\alpha\right]\, F^N_\beta\right) - \omega_t\left(\mathbb{H}^*\left[F^N_\alpha\right]\right)\omega_t\left(F^N_\beta\right)$$

$$\omega_t\left(\mathbb{H}^*\left[F^N_\alpha\right]\, F^N_\beta\right) \approx \sum_{\mathrm{s}} q^\alpha_{\mathrm{s}}\, m_{\eta_{\mathrm{s}}}(t)\, \omega_t\left(F^N_{\mu_{\mathrm{s}}} F^N_\beta\right)$$

  - Which can be rewritten as $\omega_t\left(\mathbb{H}^*\left[F^N_\alpha\right]\, F^N_\beta\right) \mapsto \sum_{\mathrm{s}} Q_{\alpha\mu_{\mathrm{s}}}(t) C^N_{\mu_{\mathrm{s}}\beta}(t)$

Buonaiuto et al. PRL **127**, 133601 (2021)

Dynamics of quantum fluctuations

- Diffusion term
  - Given by $$\frac{1}{N}\sum_{\mu,\nu}\frac{a_{\mu\nu}}{2}\left[\left[V_\mu, F_\alpha^N F_\beta^N\right], V_\nu\right] = \sum_{\mu,\nu}\frac{a_{\mu\nu}}{2}\left[\left[F_\mu^N, F_\alpha^N F_\beta^N\right], F_\nu^N\right]$$

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Dynamics of quantum fluctuations

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  - Note that
$$\left[\left[F_\mu^N, F_\alpha^N F_\beta^N\right], F_\nu^N\right] = \left[\left[F_\mu^N, F_\alpha^N\right], F_\nu^N\right] F_\beta^N + F_\alpha^N \left[\left[F_\mu^N, F_\beta^N\right], F_\nu^N\right]$$
$$+\left[F_\alpha^N, F_\nu^N\right]\left[F_\mu^N, F_\beta^N\right] + \left[F_\mu^N, F_\alpha^N\right]\left[F_\beta^N, F_\nu^N\right]$$

  - And thus $$\omega_t\left(\left[\left[F_\mu^N, F_\alpha^N F_\beta^N\right], F_\nu^N\right]\right) \mapsto s_{\alpha\nu}(t)s_{\mu\beta}(t) + s_{\mu\alpha}(t)s_{\beta\nu}(t)$$

$$\frac{1}{N}\sum_{\mu,\nu}\frac{a_{\mu\nu}}{2}\omega_t\left(\left[\left[V_\mu, F_\alpha^N F_\beta^N\right], V_\nu\right]\right) \mapsto -\left[s(t)\, a\, s(t)\right]_{\alpha\beta}$$

Buonaiuto et al. PRL **127**, 133601 (2021)

Correlations in time crystals

- Boundary time crystal

$$H_N = \frac{\Omega}{2} \sum_{k=1}^{N} \sigma_{\mathrm{x}}^{(k)} \qquad J = \sqrt{\frac{\gamma}{2N}} \sum_{k=1}^{N} \sigma_{-}^{(k)}$$

  - Initial state $|\uparrow_{\mathrm{all}}\rangle$
  - Quantum fluctuations $F_{\mathrm{x}}^{N} \mapsto x \qquad F_{\mathrm{y}}^{N} \mapsto p$

FC et al. PRA **105**, L040202 (2022)

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$$F_{\mathrm{x}}^{N} \mapsto x \qquad F_{\mathrm{y}}^{N} \mapsto p$$

- Normal mode evolution

$$\mathbb{L}^*[O] = J^\dagger(t) O J(t) - \frac{1}{2}\left\{J^\dagger(t)J(t), O\right\}$$

$$\boxed{J(t) = x - i\cos[f(t)]p}$$

Correlations in time crystals

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$$\boxed{J(t) = x - i\cos[f(t)]p}$$

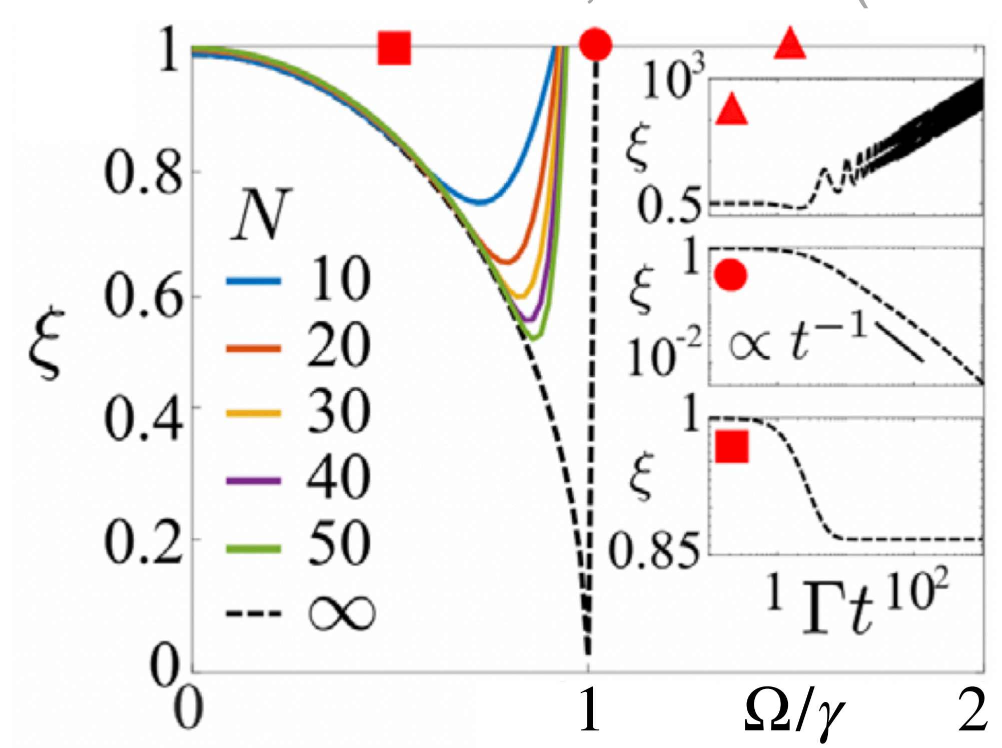
  - Stationary phase

$$J(\infty) = x - i\cos[f(\infty)]p$$

$$\Sigma(\infty) = \frac{1}{2}\begin{pmatrix} \sqrt{1-\Omega^2/\gamma^2} & 0 \\ 0 & 1/\sqrt{1-\Omega^2/\gamma^2} \end{pmatrix}$$

Spin squeezing

Buonaiuto et al. PRL **127**, 133601 (2021)



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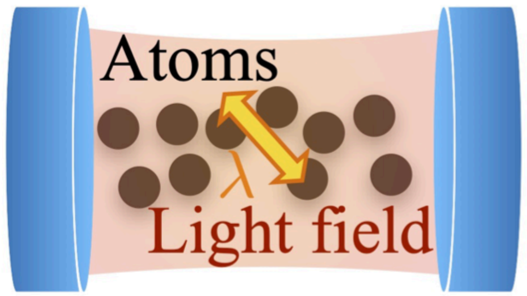
Spin-boson models

- Interacting light-matter system

Mattes et al. PRA **108**, 062216 (2023)



$$H_N = \Omega\, S_x + \frac{\lambda}{\sqrt{N}}\left(a^\dagger S_- + a S_+\right)$$

$$J = \sqrt{\kappa} a$$

Spin-boson models

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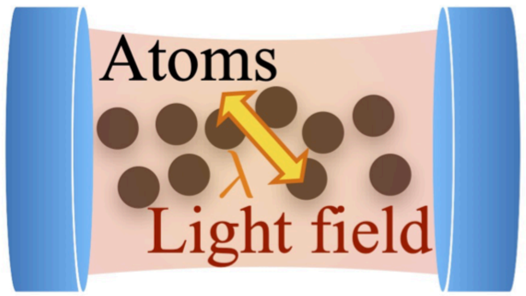
Mattes et al. PRA **108**, 062216 (2023)

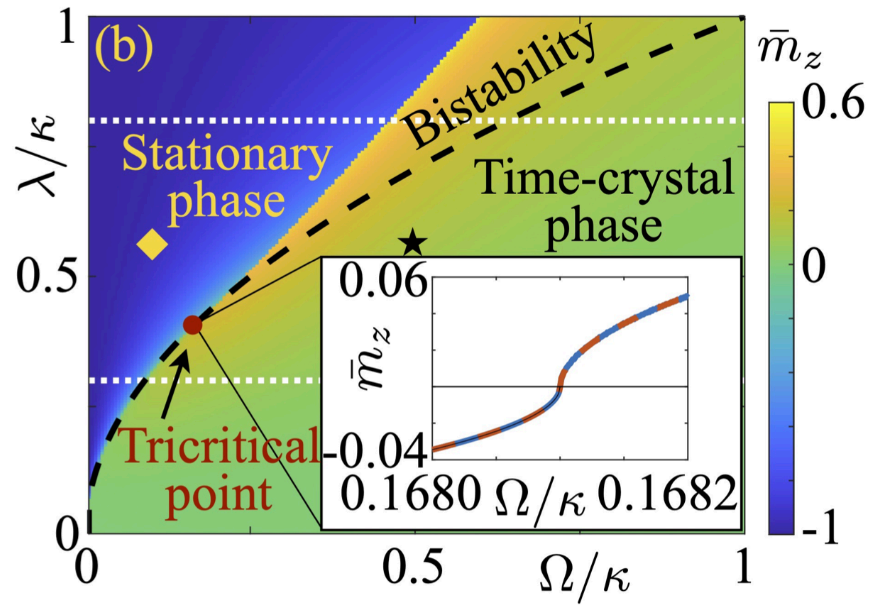


$$H_N = \Omega\, S_x + \frac{\lambda}{\sqrt{N}}\left(a^\dagger S_- + a S_+\right)$$

$$J = \sqrt{\kappa} a$$

- Mean-field phase diagram (exact)

$$\begin{cases} \{m_\mu^N\}_\mu \\ a/\sqrt{N} \end{cases}$$



Spin-boson models

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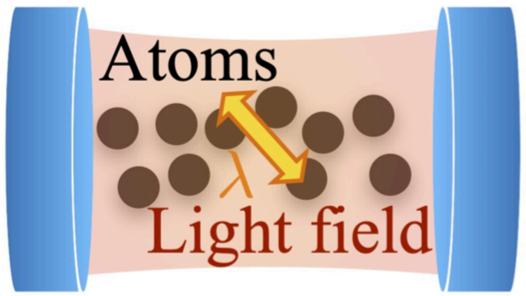
Mattes et al. PRA **108**, 062216 (2023)



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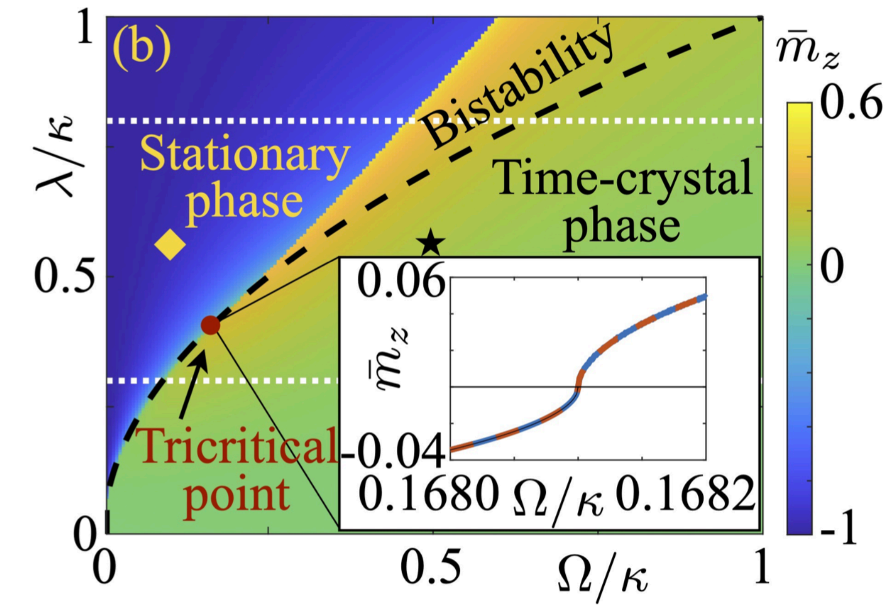
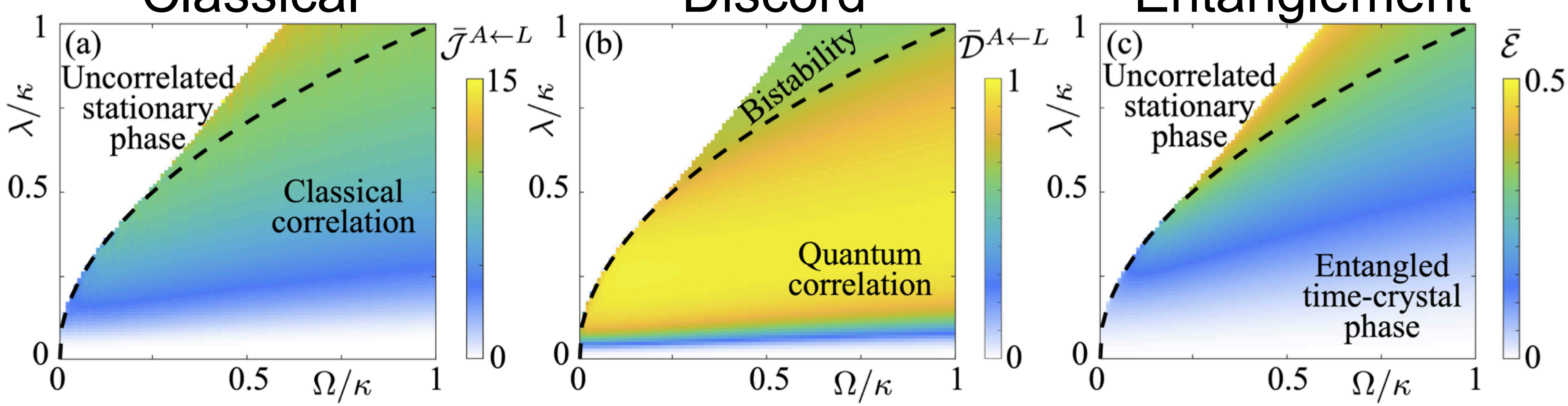


- Two-mode covariance matrix

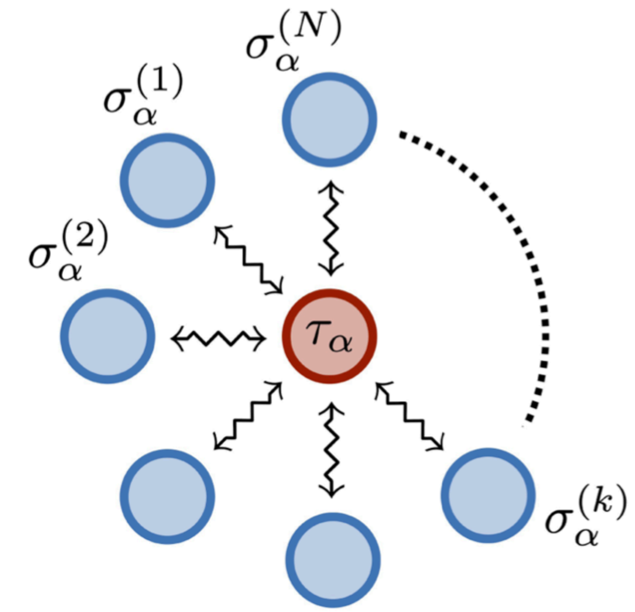
Classical Discord Entanglement



Non-Gaussian quantum fluctuations

- Central spin systems

$$H_N \propto \frac{g}{\sqrt{N}}\left(\tau_+ S_- + \tau_- S_+\right)$$



- Reminiscent of spin-boson models

$$H_N \propto \frac{\lambda}{\sqrt{N}}\left(a^\dagger S_- + a S_+\right)$$

Dynamics preserves Gaussianity

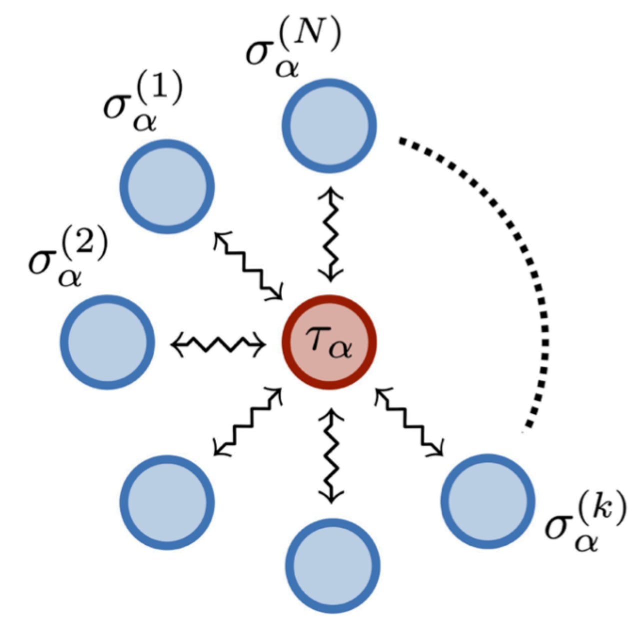
$$H \propto \lambda\left(a^\dagger b + a b^\dagger\right)$$

FC, PRL **131**, 227102 (2023)

Non-Gaussian quantum fluctuations

- Central spin systems

$$H_N \propto \frac{g}{\sqrt{N}} \left(\tau_+ S_- + \tau_- S_+\right)$$



- Reminiscent of spin-boson models

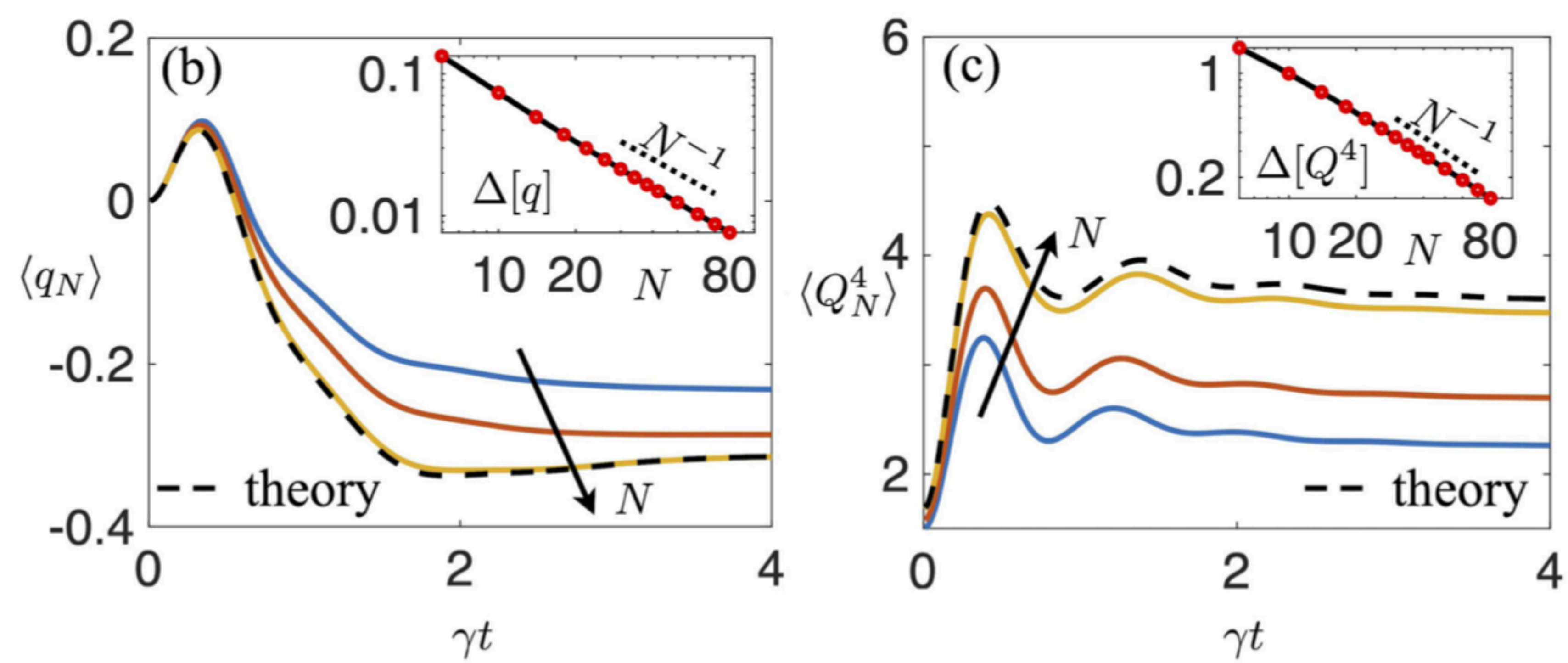
$$H_N \propto \frac{\lambda}{\sqrt{N}} \left(a^\dagger S_- + a S_+\right) \longrightarrow$$

Dynamics preserves Gaussianity

$$H \propto \lambda \left(a^\dagger b + a b^\dagger\right)$$

- Emergent fluctuation dynamics is non-Gaussian

$$H_N \propto \frac{g}{\sqrt{N}} \left(\tau_+ S_- + \tau_- S_+\right)$$
$$\downarrow$$
$$H \propto g \left(\tau_+ b + \tau_- b^\dagger\right)$$



FC, PRL **131**, 227102 (2023)

Non-Gaussian quantum fluctuations

- Non-clustering state and abnormal fluctuations

$$\omega(O) = \langle S, m | O | S, m \rangle \qquad \lim_{N\to\infty} \frac{m}{N} \neq S$$

- Not a clustering state

$$\omega\left(\left[m_x^N - \omega(v_x)\right]^2\right) \not\longrightarrow 0 \qquad \omega\left(F_x^N F_x^N\right) \longrightarrow \infty$$

Benatti et al. NJP **26**, 013057 (2024)

Non-Gaussian quantum fluctuations

- Non-clustering state and abnormal fluctuations

$$\omega(O) = \langle S, m \,|\, O \,|\, S, m \rangle \qquad \lim_{N\to\infty} \frac{m}{N} \neq S$$

- Not a clustering state

$$\omega\left(\left[m_{\mathrm{x}}^{N} - \omega(v_{\mathrm{x}})\right]^{2}\right) \not\longrightarrow 0 \qquad \omega\left(F_{\mathrm{x}}^{N} F_{\mathrm{x}}^{N}\right) \longrightarrow \infty$$

- Abnormal fluctuations (need to modify the scaling)

$$F_{\pm}^{N} = \frac{S_{\pm}}{N} \qquad F_{\mathrm{z}}^{N} = S_{\mathrm{z}} \qquad \Longrightarrow \qquad \left[F_{\mathrm{z}}^{N}, F_{\pm}^{N}\right] = \pm F_{\pm}^{N} \qquad F_{+}^{N} F_{-}^{N} \to \mathbf{1}$$

- Heisenberg algebra on the ring

$$F_{\pm}^{N} \to e^{\pm i\theta} \qquad F_{\mathrm{z}}^{N} \to p_{\theta}$$

$$\left[\theta, p_{\theta}\right] = i$$

Derivation of the Josephson junction Hamiltonian

$$H = p_{\theta}^{2} + J\cos\theta$$

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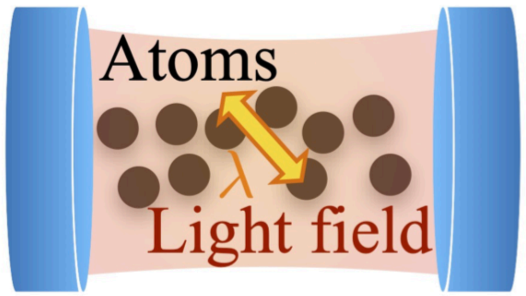
Conclusions

Open quantum systems with long-range interactions

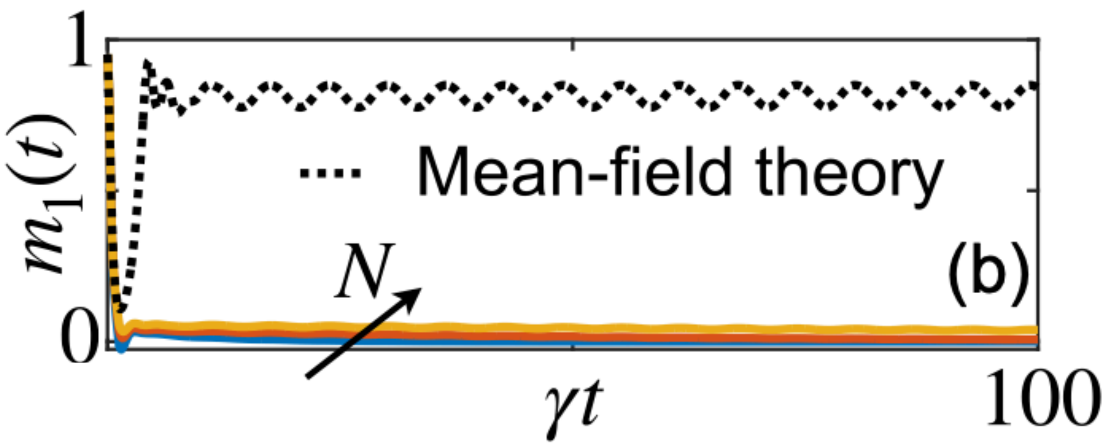


- Relevant for cavity QED, clouds of atoms, etc
- Models for non-equilibrium phase transitions

Exactness of mean-field approximation

- Strong long-range regime
- Proof gives a clear interpretation

$$\mathscr{E}_N(t) \sim \frac{e^{tC_1}}{N}$$



Quantum fluctuations (central limit theorem)

- Even collective models show quantum and classical correlations
- Generically captured by quantum fluctuation operators

$$\lim_{N\to\infty} \omega\left(e^{ir_1 F_x^N + ir_2 F_y^N}\right) = e^{-(r_1^2+r_2^2)/4} = \langle 0 | e^{ir_1 x + ir_2 p} | 0 \rangle$$

Thank you very much for your attention