Circuits is all you need

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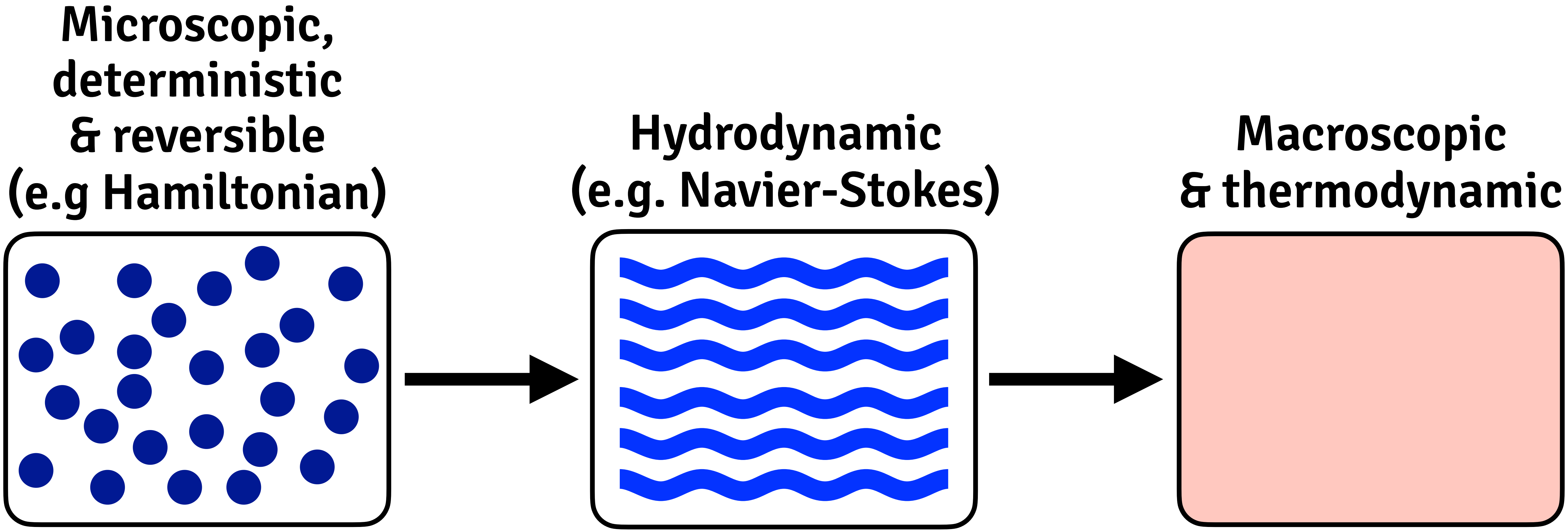
With: Sun Woo Kim, Friedrich Hübner & Benjamin Doyon (KCL)

"Circuits as a simple platform for the emergence of hydrodynamics in deterministic chaotic many-body systems", **arXiv:2503.08788**

£: EPSRC The Leverhulme Trust CQNE Centre for the Mathematics and Theoretical Physics of Quantum Non-Equilibrium Systems Machine Learning in Science School of Physics & Astronomy and Faculty of Science

MOTIVATION



Microscopic, deterministic & reversible (e.g Hamiltonian)

Hydrodynamic (e.g. Navier-Stokes)

Macroscopic & thermodynamic

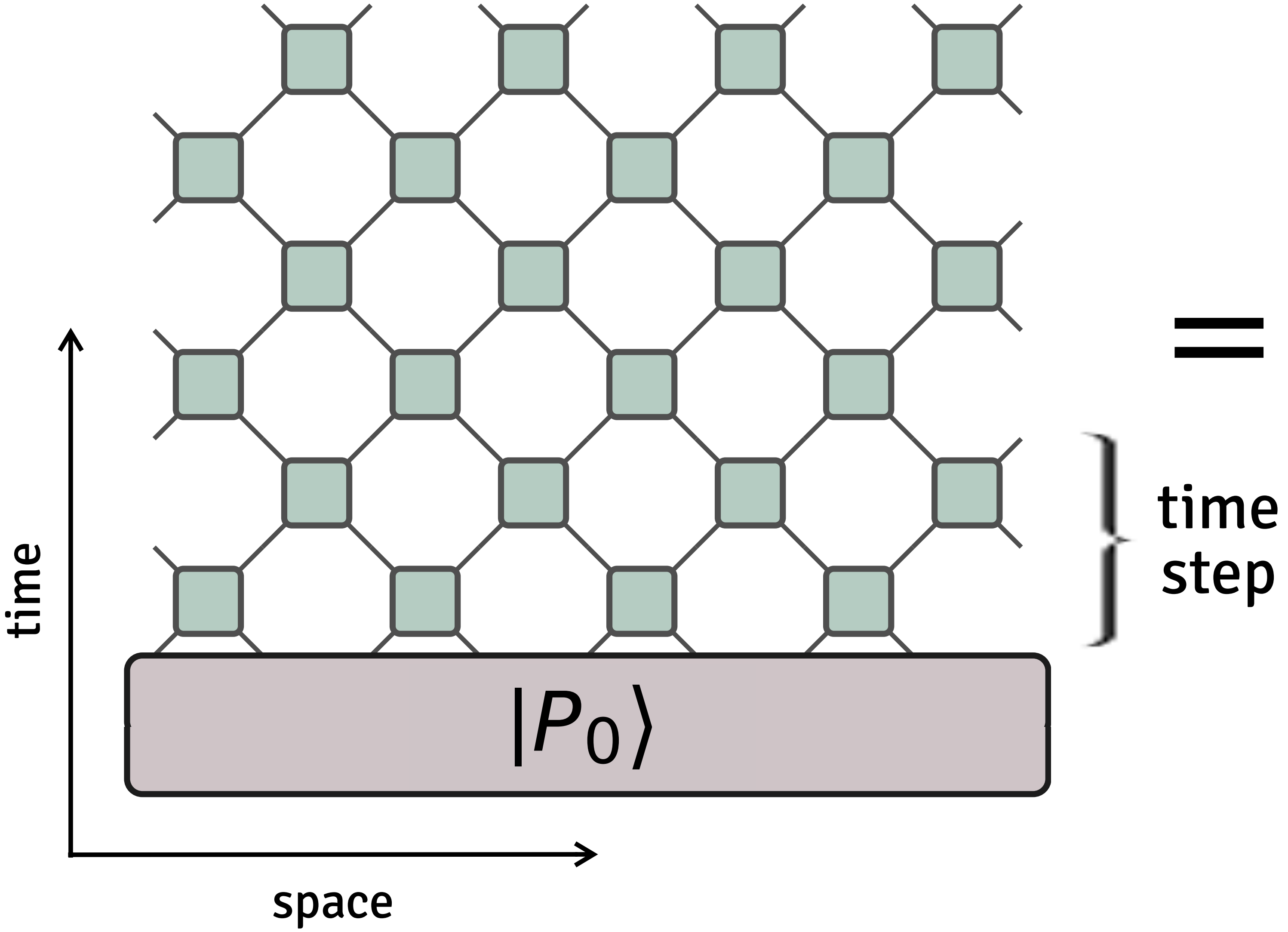
dimensional reduction & emergence of irreversibility

Poorly understood & many practical issues… Exception → GHD {Castro Alvaredo-et-al 2016, Bertini-et-al 2016, Doyon 2020 …}

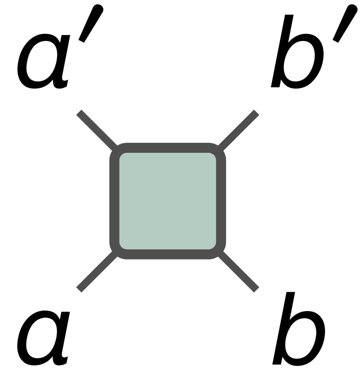
minimal general platform? → “chaotic” classical permutation circuits (aka deterministic cellular automata)

Classical deterministic circuits

$a \in \{0, 1 \cdots d-1\}$



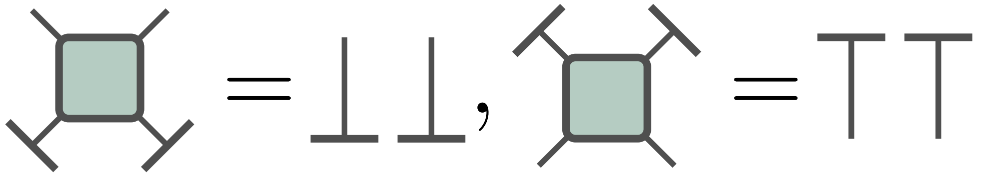
$= |P_t\rangle$



permutation $\#\sigma = (d^2)!$

$\sigma : (a, b) \to (a', b')$

Cf. **RND** permutation circuits
{Bertini-Klobas-Kos-Malz 2024}

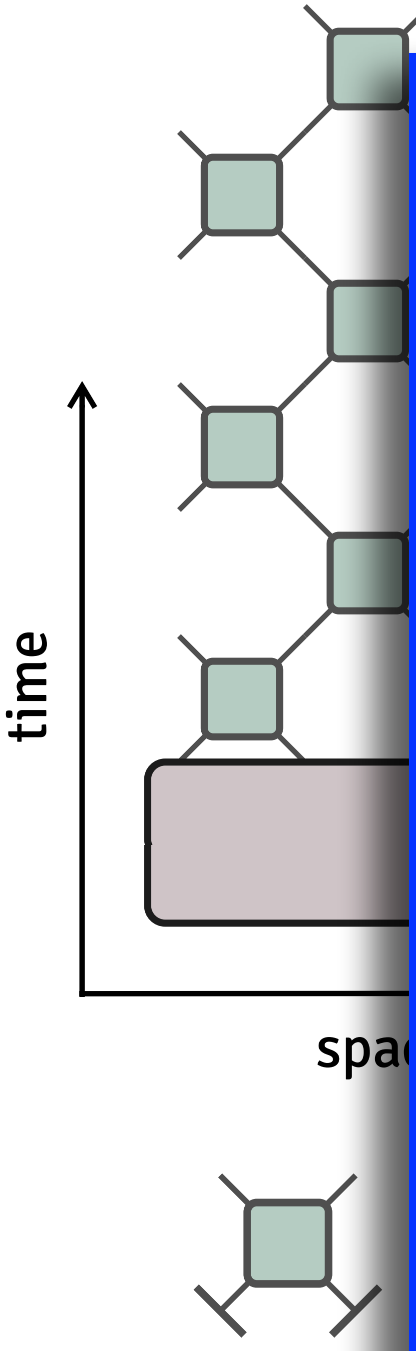


$\Rightarrow$ **flat state =** $\perp\perp\cdots\perp = \langle -|$

(bi) invariant

Classical deterministic circuits

$a \in \{0, 1 \cdots d-1\}$



⇒ **fla**

(bi)

$d^2)!$

$')$

cuits

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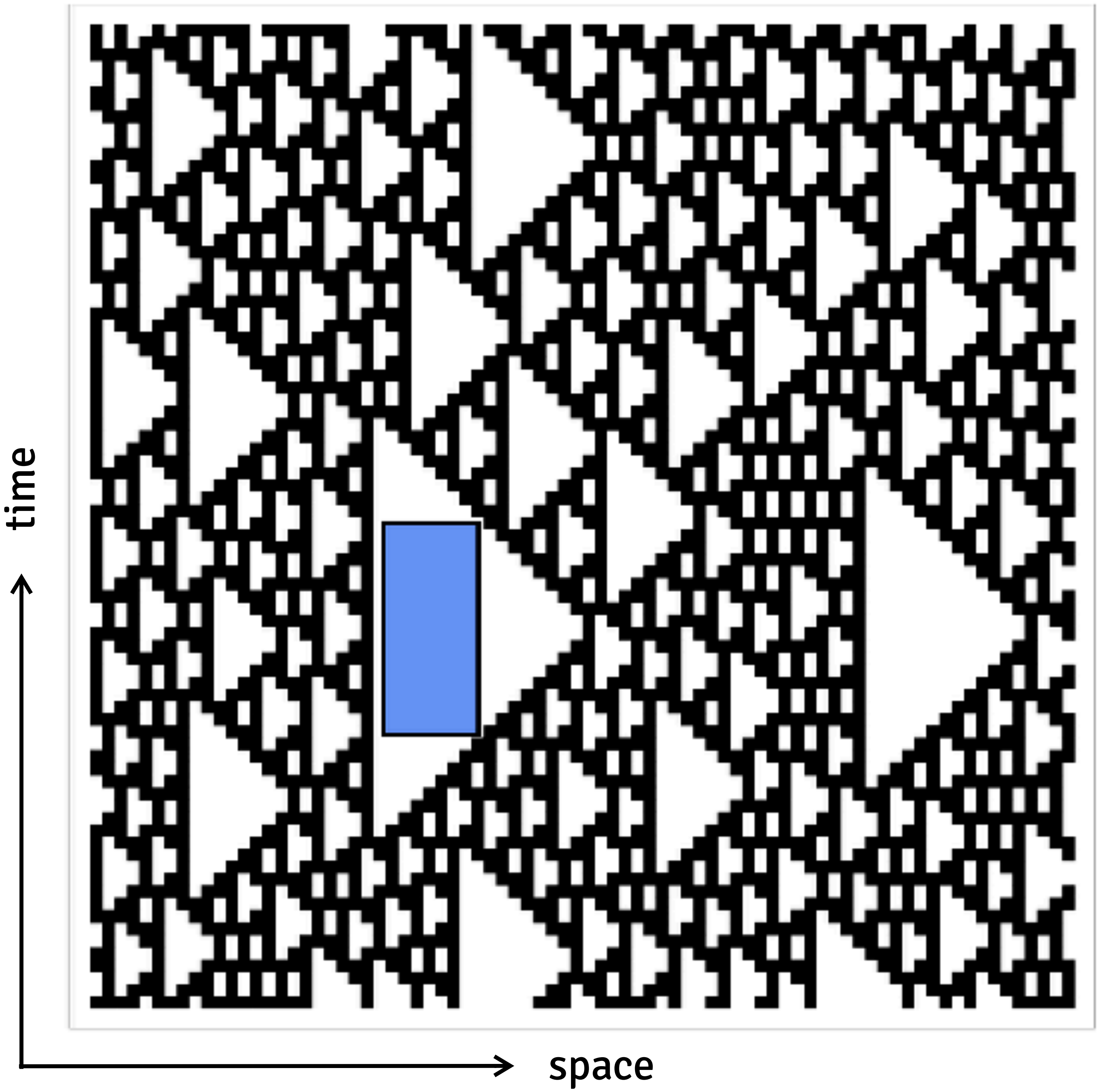
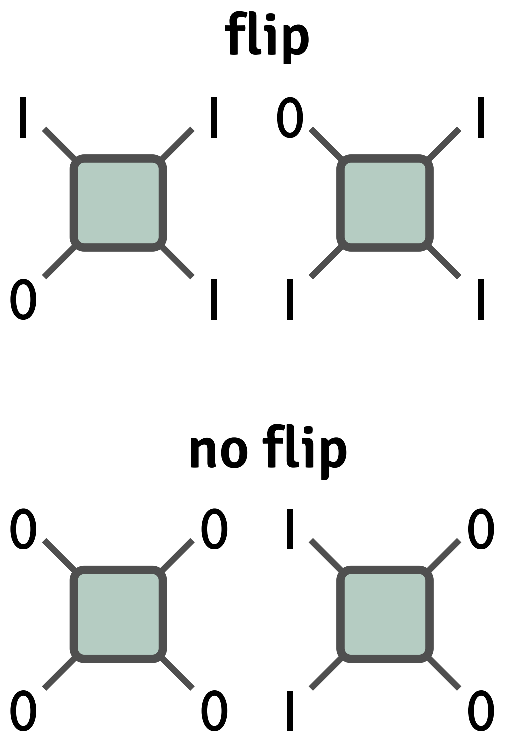
Eg. East circuit

{Klobas-De Fazio-JPG, **PRE 2024 & JPA 2024**}

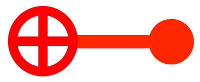
$d = 2$



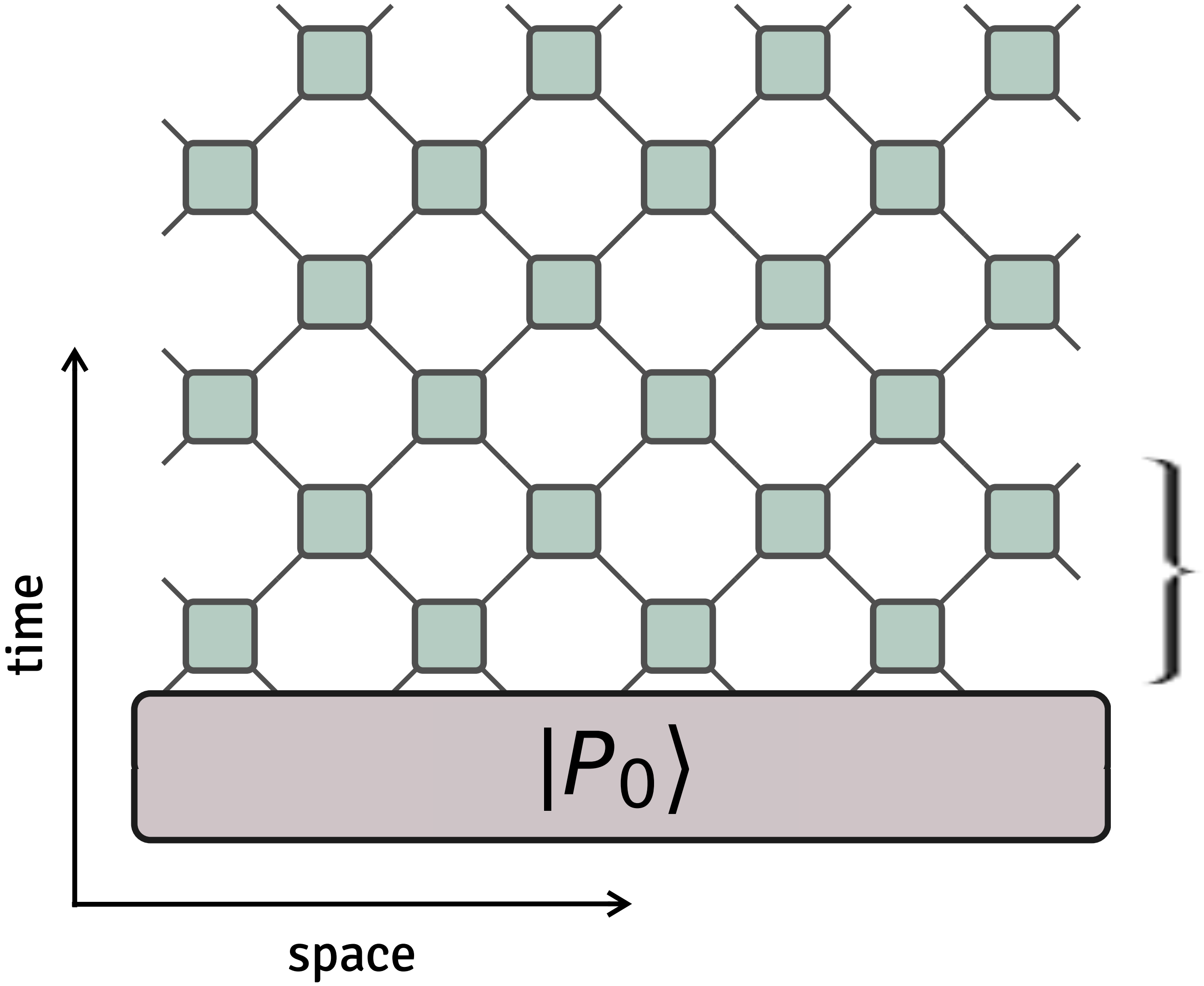


East KCM rules

CNOT

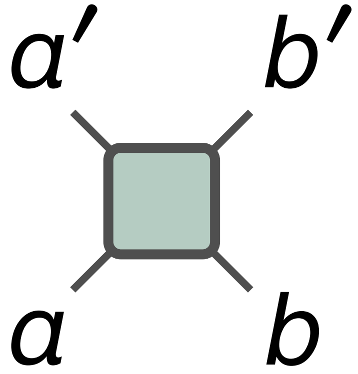


Classical deterministic circuits



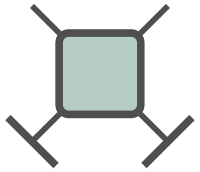
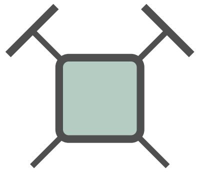
$$= |P_t\rangle$$

$$a \in \{0, 1 \cdots d-1\}$$



permutation $\#\sigma = (d^2)!$

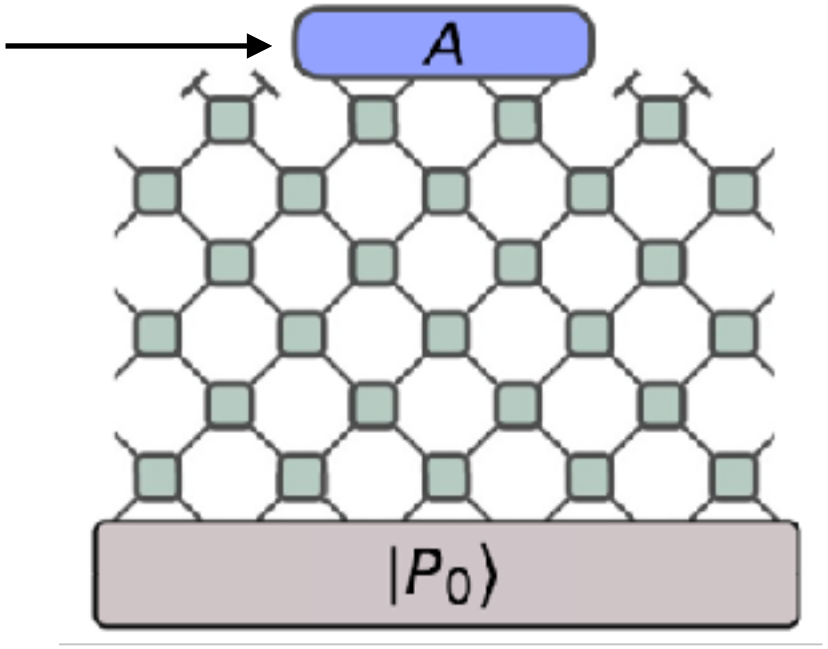
$$\sigma : (a, b) \to (a', b')$$

 $= \perp\perp$,  $= \top\top$

⇒ **flat state =** $\perp \perp \cdots \perp = \langle -|$

(bi) invariant

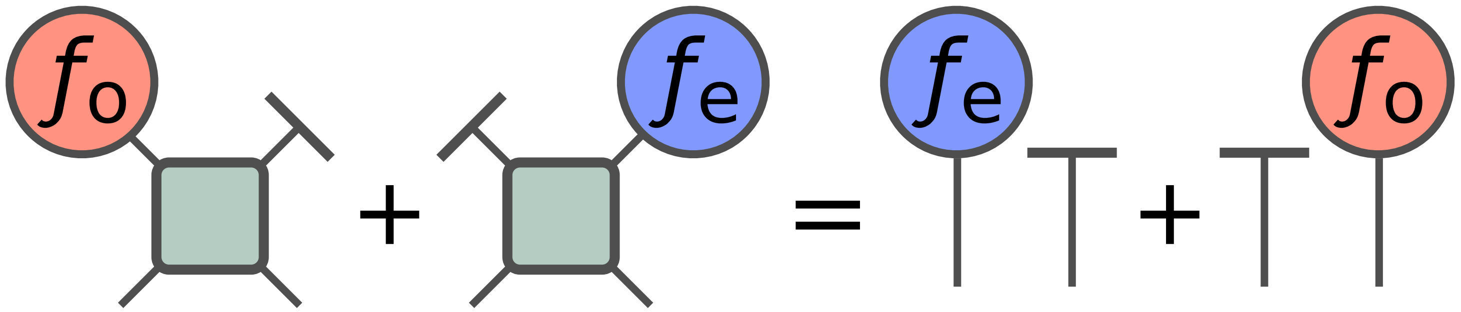
$$\langle A_t \rangle = \underbrace{\langle - | A}_{\langle A|} U_t | P_0 \rangle =$$



Conserved quantities

$$\langle F| = \sum_{j\,\text{odd}} \overset{\text{period} = m}{\mu^j} \underbrace{\left(\langle F_j^{\text{o}}| + \langle F_j^{\text{e}}| \right)}_{\text{support} = l} \xrightarrow{} \text{conserved} = n \text{ steps}$$

$l, m, n = 1$:



Can get all local CQs up to $l, m, n = 10$ (not quasi-local)

For $d = 2$: either fully chaotic (i.e. **no CQs**, e.g. East) or **integrable**

For $d = 3$: gates with one $\underbrace{\text{1-local CQ}}_{\textbf{Gibbs state = product}} \approx 2000$, out of $(3^2)! = 362880$

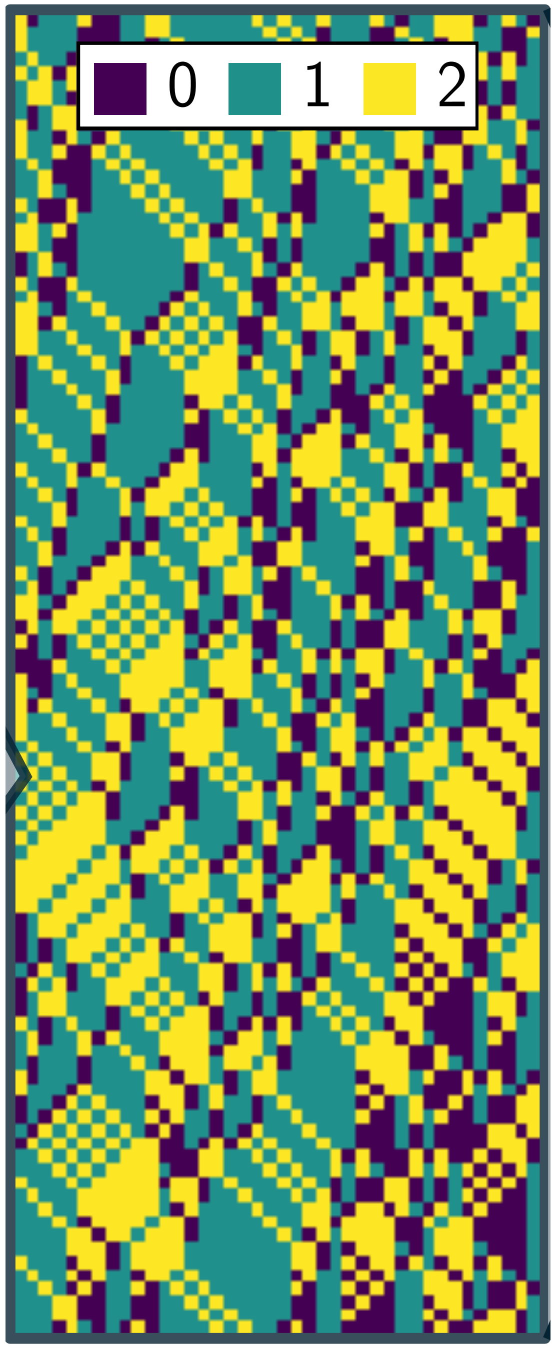
Conserved quantities

Example: $(d, \sigma) = (3, 996)$

$|00\rangle \to |00\rangle \quad |01\rangle \to |01\rangle \quad |02\rangle \to |10\rangle$

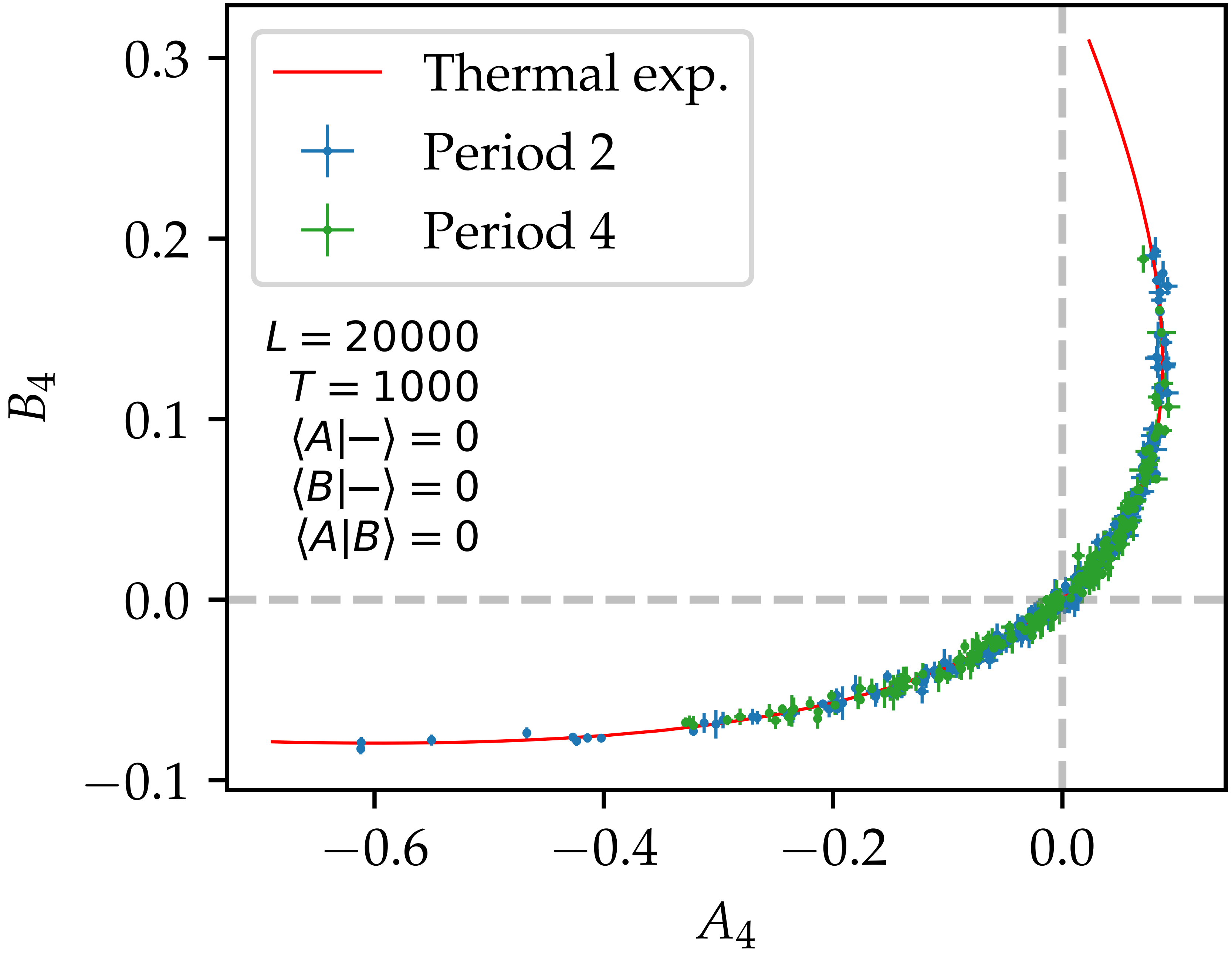
$|10\rangle \to |12\rangle \quad |11\rangle \to |11\rangle \quad |12\rangle \to |21\rangle$

$|20\rangle \to |02\rangle \quad |21\rangle \to |20\rangle \quad |22\rangle \to |22\rangle$



CQ: $\langle f_{\mathrm{e}}| = \langle 1|,\ \langle f_{\mathrm{o}}| = -\langle 0|$

$$|P_\beta\rangle = \frac{1}{Z_\beta}\sum_{\boldsymbol{\alpha}} e^{-\beta F}|\boldsymbol{\alpha}\rangle \to \langle A|P_\beta\rangle$$



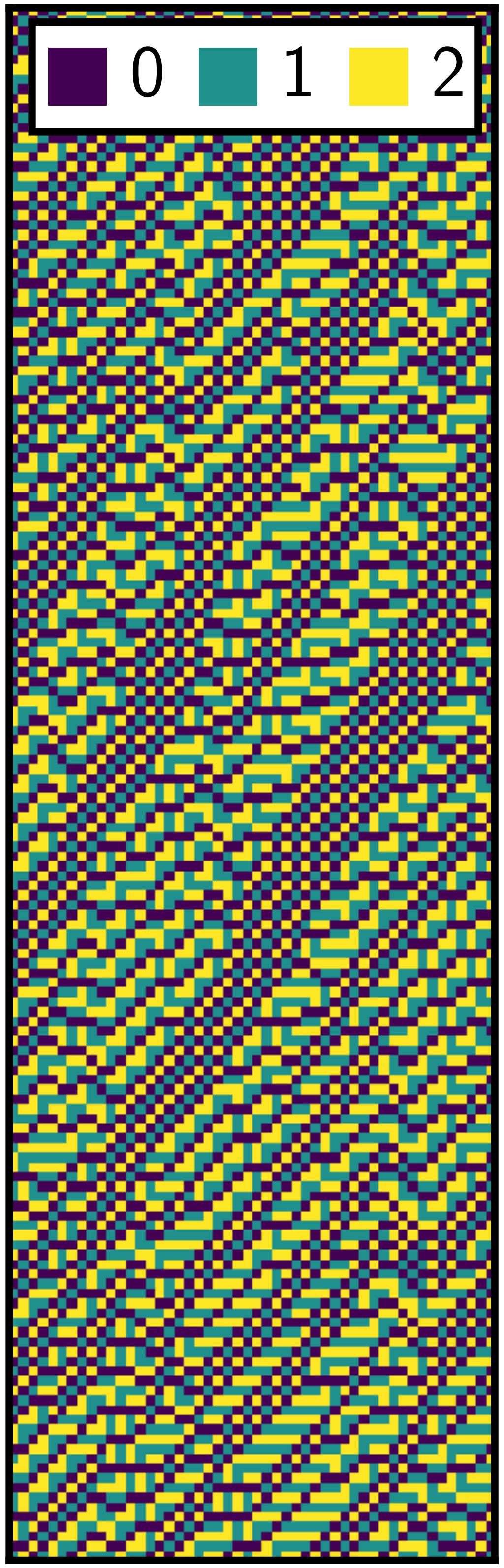
Conserved quantities

Example: $(d, \sigma) = (3, 229117)$

$|00\rangle \to |12\rangle$
$|10\rangle \to |02\rangle$
$|20\rangle \to |01\rangle$

$|01\rangle \to |22\rangle$
$|11\rangle \to |00\rangle$
$|21\rangle \to |11\rangle$

$|02\rangle \to |20\rangle$
$|12\rangle \to |10\rangle$
$|22\rangle \to |21\rangle$



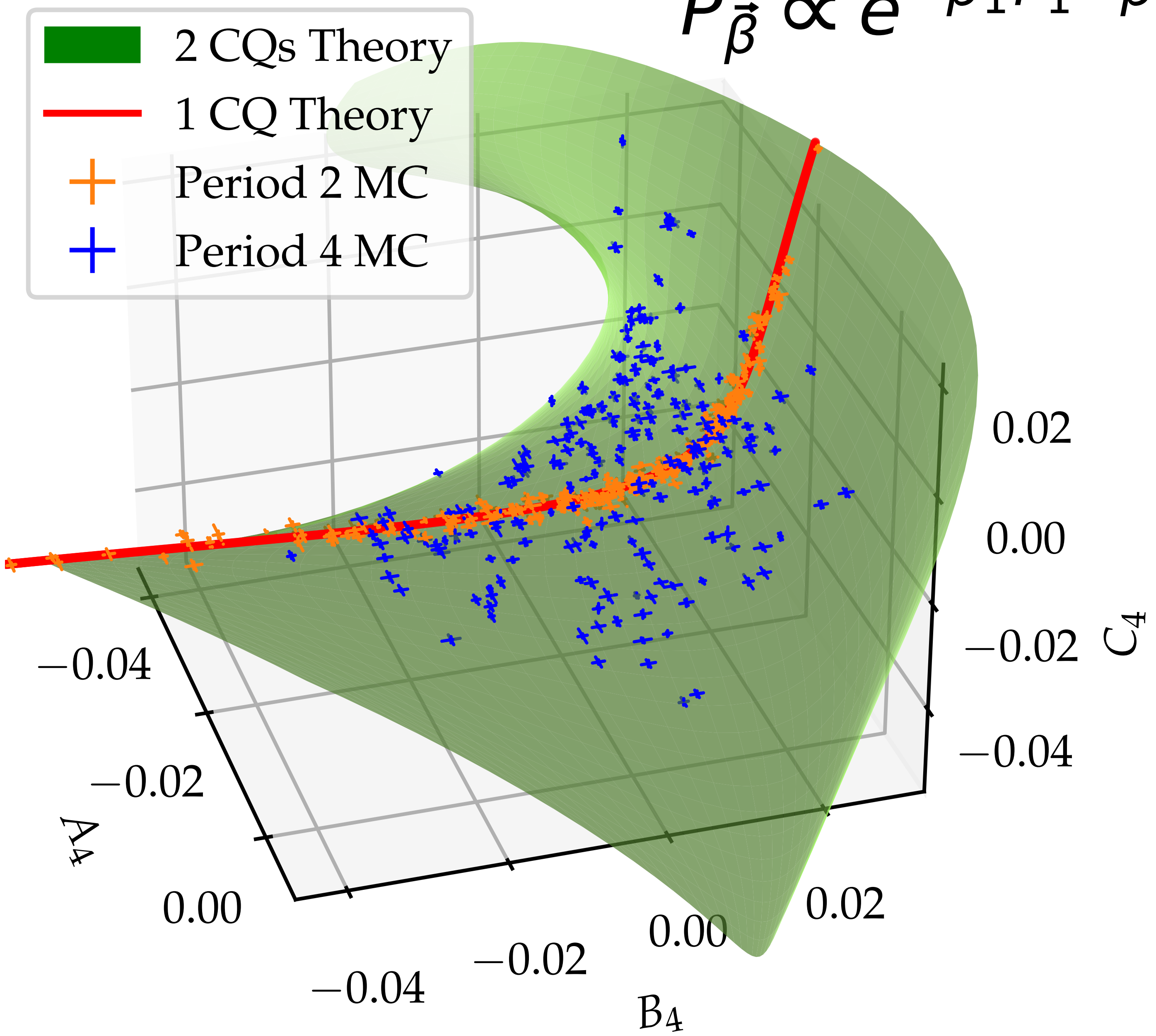
CQs: $\langle f_o^{(1)}| = \langle 1| + \langle 2|$, $\langle f_e^{(1)}| = \langle 0| + 2\langle 1| + \langle 2|$

$n = m = 1$

$\langle f_o^{(2)}| = \langle 0| - \langle 1| + \langle 2|$, $\langle f_e^{(2)}| = \langle 1| + 2\langle 2|$

$n = m = 2$

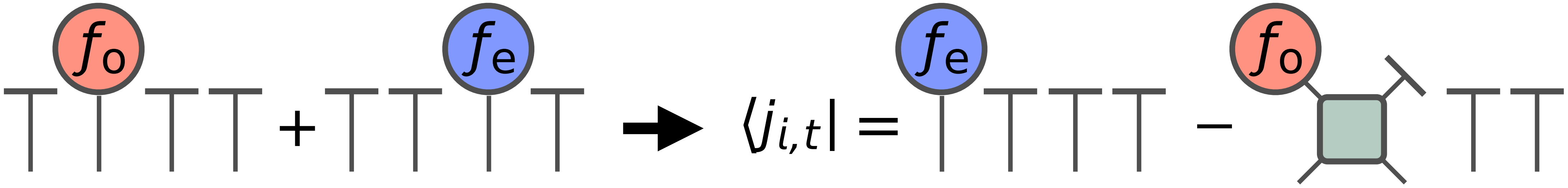
$$P_{\vec{\beta}} \propto e^{-\beta_1 F_1 - \beta_2 F_2}$$



Euler scale HD

micro. cont. eqs. $$\frac{\langle q_{i,t+n}| - \langle q_{i,t}|}{n} = -\frac{\langle j_{i+m,t}| - \langle j_{i,t}|}{m} \longrightarrow \boxed{\partial_t \vec{q} = -\mathbb{A}(q)\cdot \partial_x \vec{q}}$$

$l, m, n = 1:$



$\langle q_{i,t}| =$ … $+$ … $\rightarrow$ $\langle j_{i,t}| =$ … $-$ …

$$\vec{q}(\vec{\beta}) = \langle \mathbf{q}|P_{\vec{\beta}}\rangle \longrightarrow \vec{\beta}(\vec{q})$$

$$\vec{j}(\vec{\beta}) = \langle \mathbf{j}|P_{\vec{\beta}}\rangle \longrightarrow \vec{j}(\vec{q})$$

assume local equilibrium $\vec{\beta}(x,t) \rightarrow \vec{q}(x,t)$

$$\mathbb{A}_{ab} = \frac{\partial j_a}{\partial q_b}$$

$$(d,\sigma) = (3,996): \begin{pmatrix} \langle q|P_\beta\rangle \\ \langle j|P_\beta\rangle \end{pmatrix} = \pm\frac{1}{2e^\beta+1} + \frac{2}{e^\beta+2} - 1 \;\rightarrow\; j = \frac{1}{3}\left(2 - \sqrt{9q^2+16}\right)$$

Burgers eq. for $v = j'(q)$

Not possible in time-continuous systems $(\mathrm{C}Q = H \Rightarrow \partial_t \rho = 0)$

Shocks & entropy production

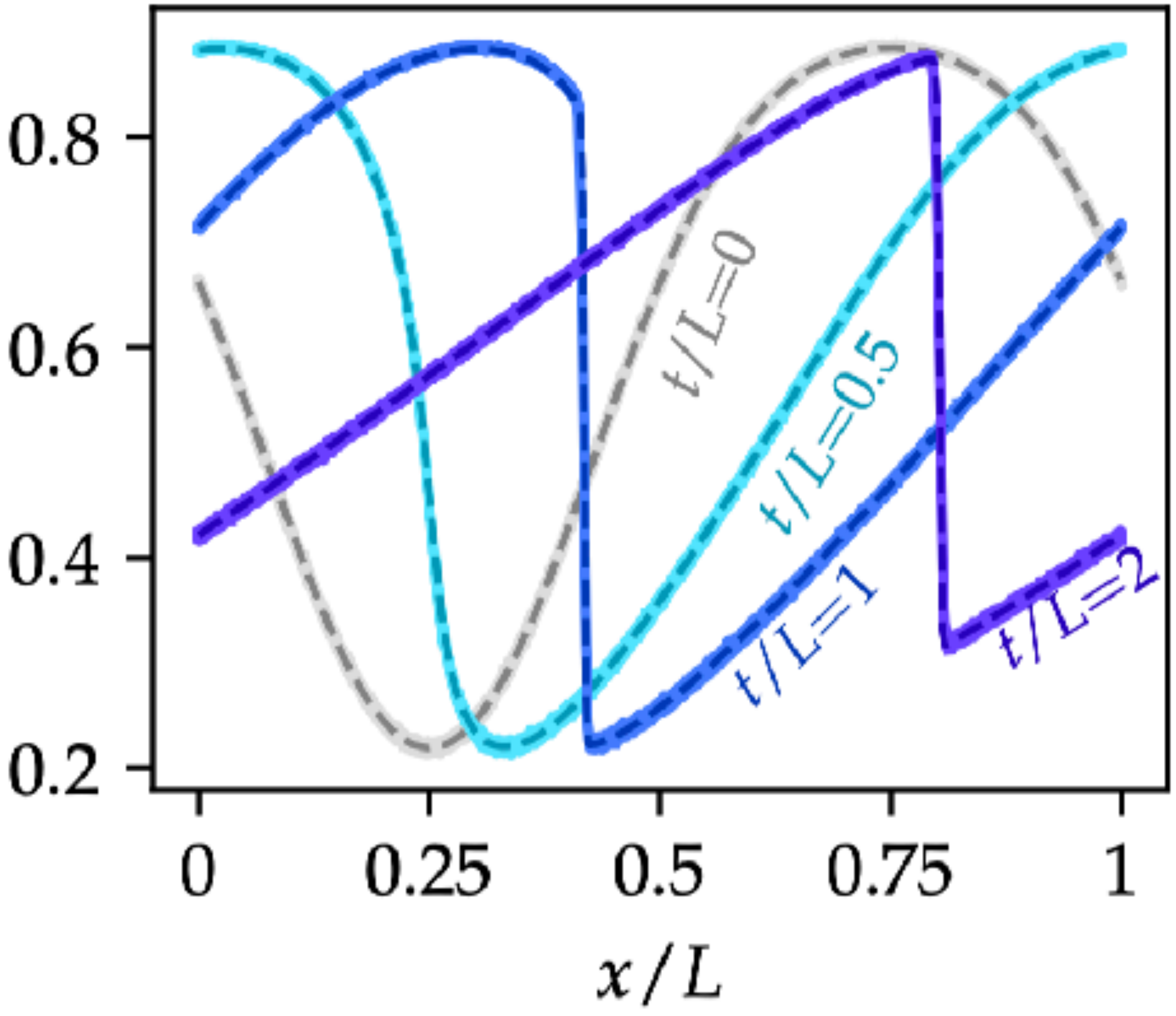
$(d, \sigma) = (3, 996)$:

$$\partial_t q = -\left(\frac{3q}{\sqrt{16 + 9q^2}}\right)\partial_x q$$

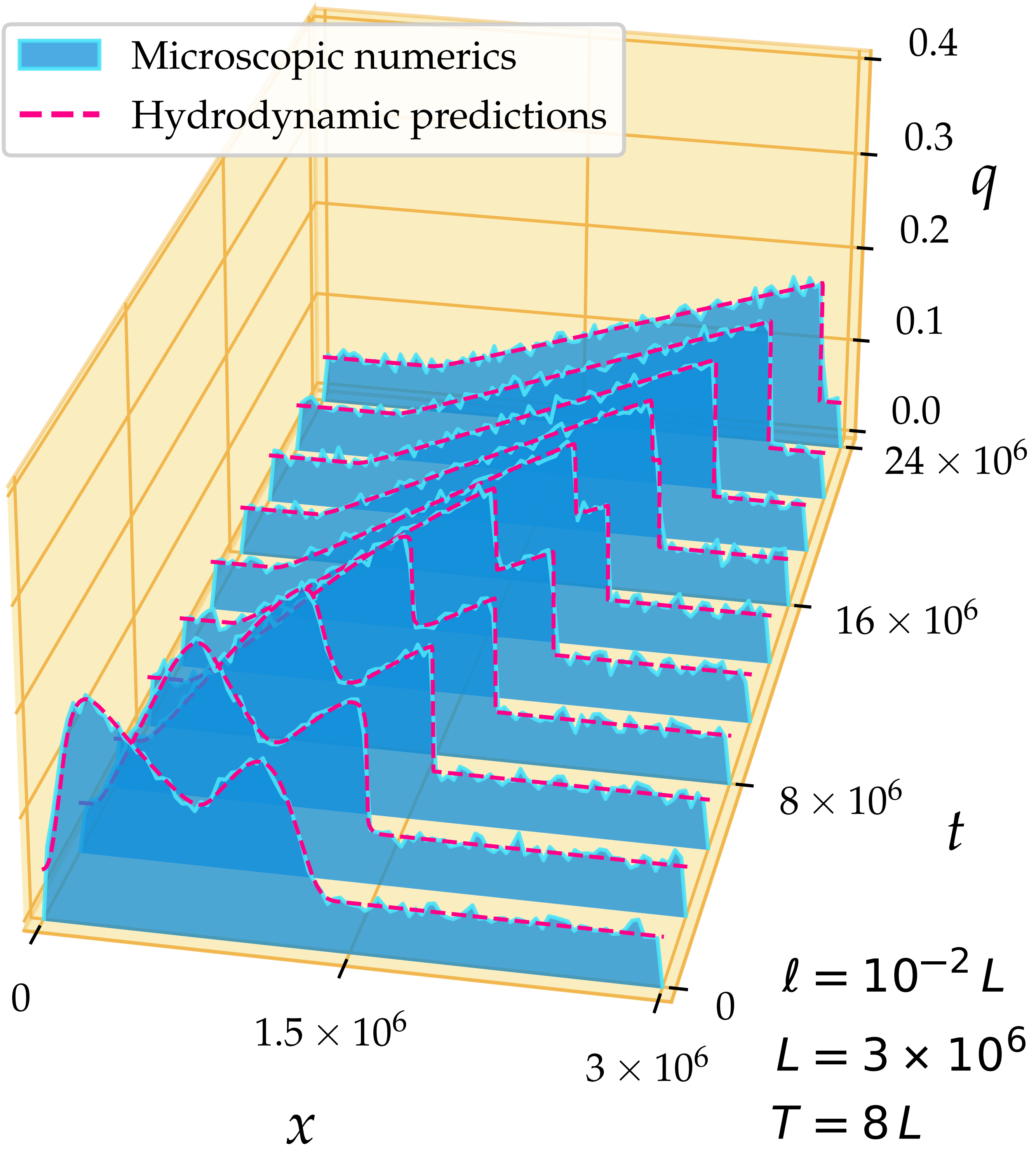
$L = 10^5$

samples $= 10^5$

$q(x, t)$



self averaging



$\ell = 10^{-2}\,L$

$L = 3 \times 10^6$

$T = 8\,L$

Shocks & entropy production

$(d, \sigma) = (3, 996)$:

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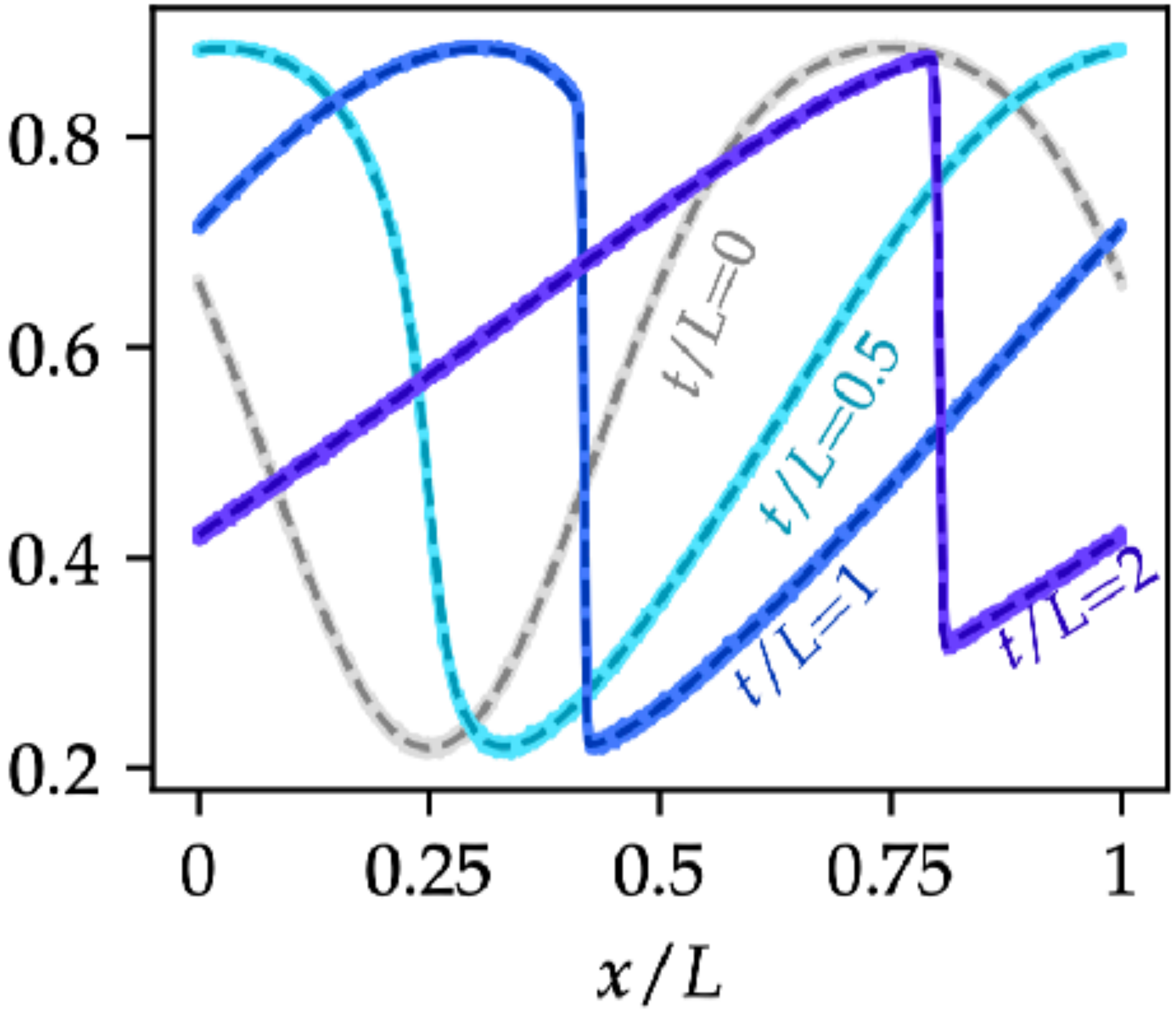
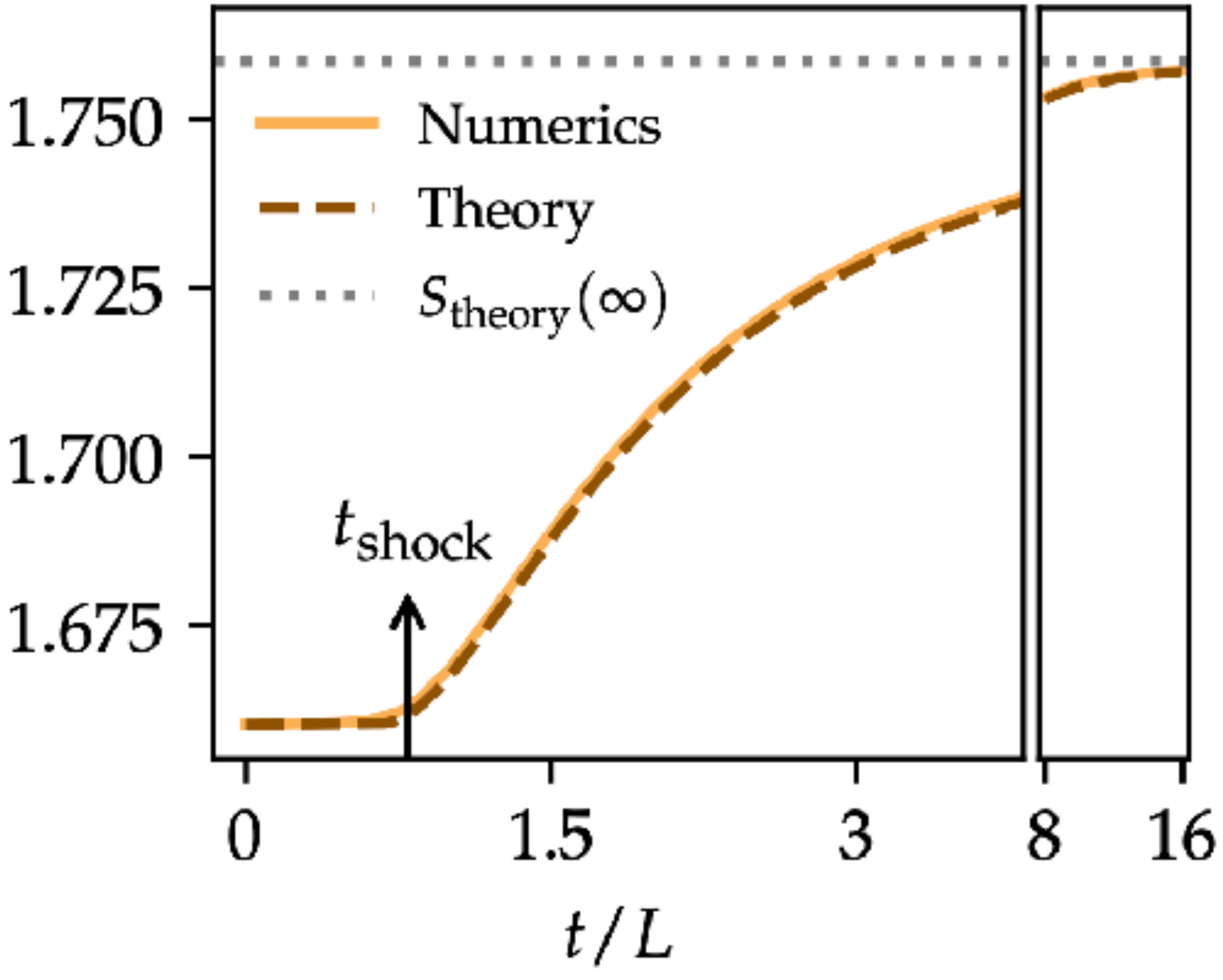


$s(t) = -L^{-1} \int_x \langle \ln P_{\beta(q(x,t))} \rangle$



Non-linear fluctuating HD

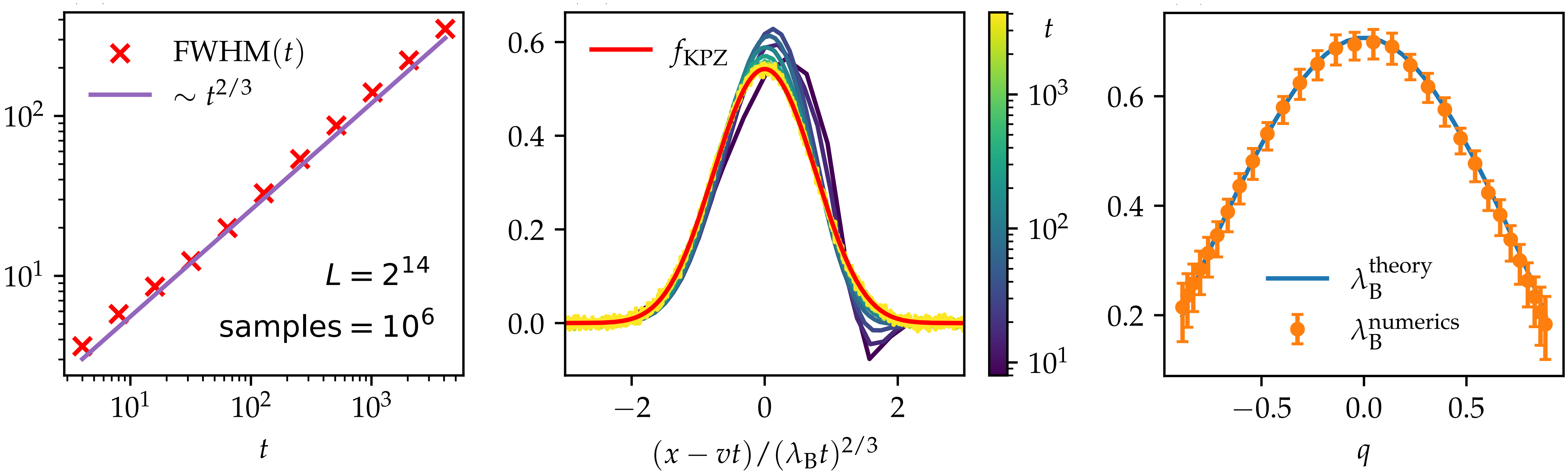
fluct. on stationary background: $q(x,t) = q_0 + \delta q(x,t) \quad q_0 = \langle q \rangle_\beta$

{Spohn 2014}

$$\partial_t \delta q + j'(q_0)\,\partial_x \delta q + \frac{1}{2} j''(q_0)\,\partial_x \delta q^2 + (\text{diffusion}) + (\text{noise}) = 0$$

$j''(q_0) \neq 0 \rightarrow$ **KPZ:** $\langle \delta q(x,t) \delta q(0,0) \rangle = \frac{\langle \delta q^2 \rangle_\beta}{(\lambda t)^{2/3}} f_{\mathrm{KPZ}}\left(\frac{x - vt}{(\lambda t)^{2/3}}\right)$ **with** $\lambda = \sqrt{2 \langle \delta q^2 \rangle_\beta}\, j''(q_0)$

$(d, \sigma) = (3, 996)$: $j''(q) \neq 0 \rightarrow$ **KPZ:**



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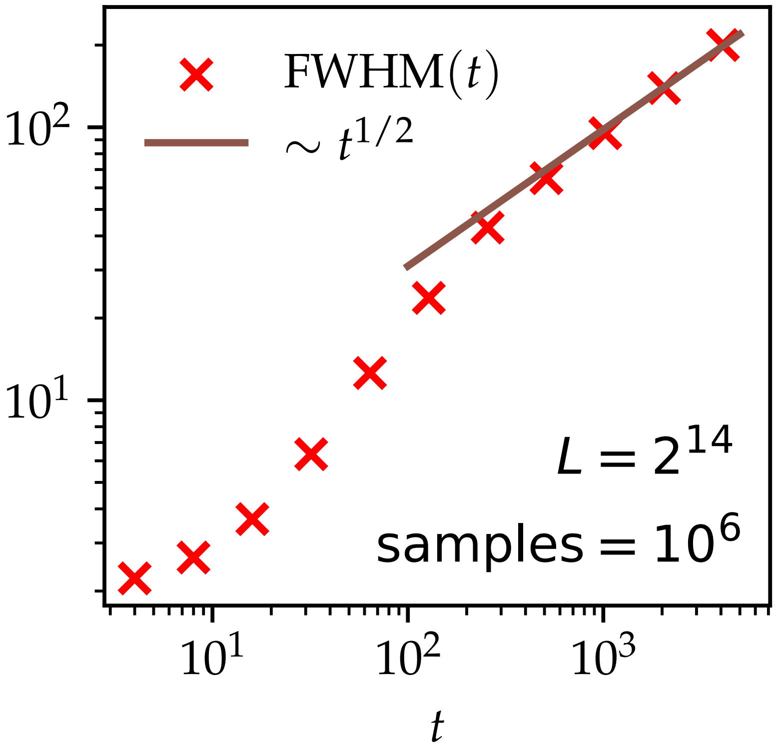
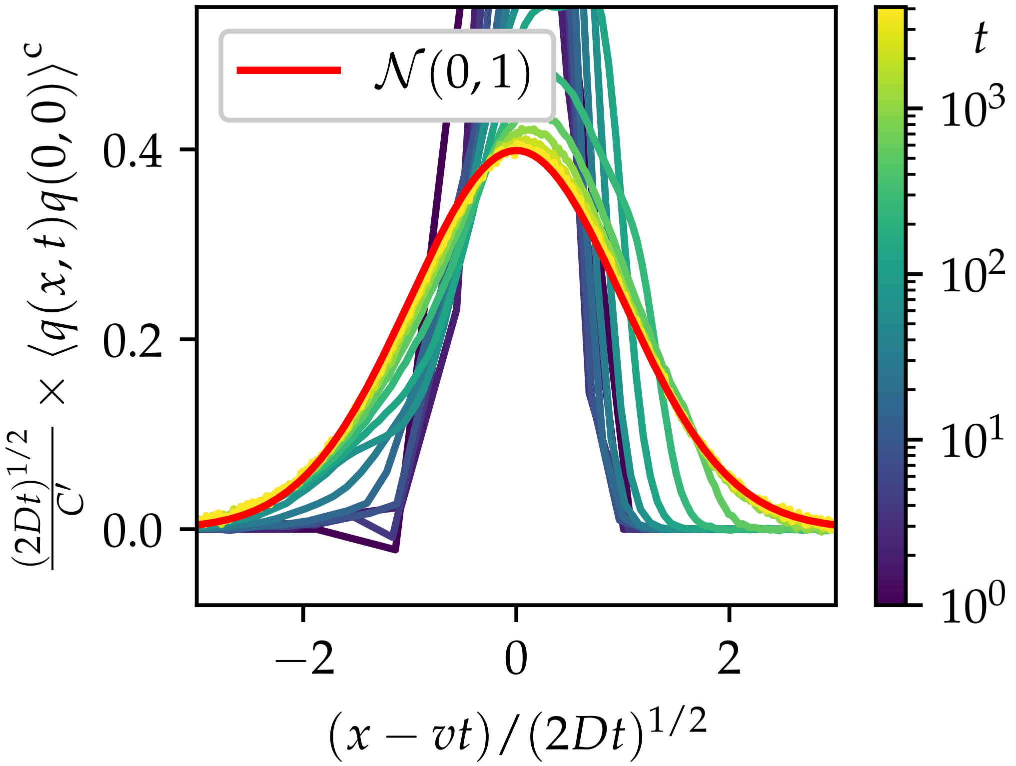
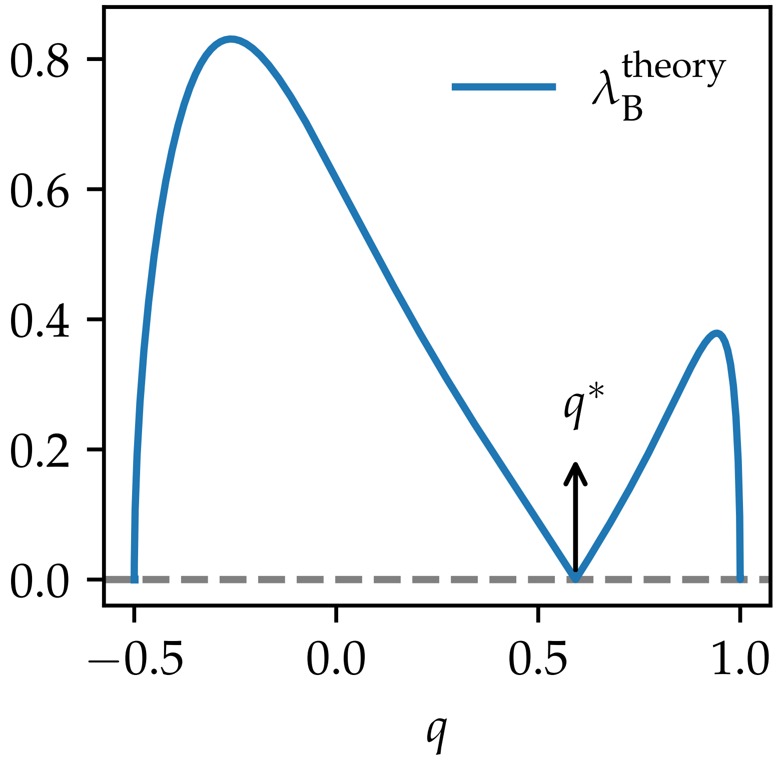
$j''(q) = 0 \rightarrow$ **diffusive**

$(d, \sigma) = (3, 1092)$: $j''(q_0 = q^*) = 0$







Fermi–Pasta–Ulam–Tsingou behaviour

Example: $(d, \sigma) = (3, 2312)$ **CQ:** $\langle f_{\mathrm{o}}| = \langle f_{\mathrm{e}}| = \langle 2|$ $j(q) = j''(q) = 0$

$|00\rangle \rightarrow |00\rangle$
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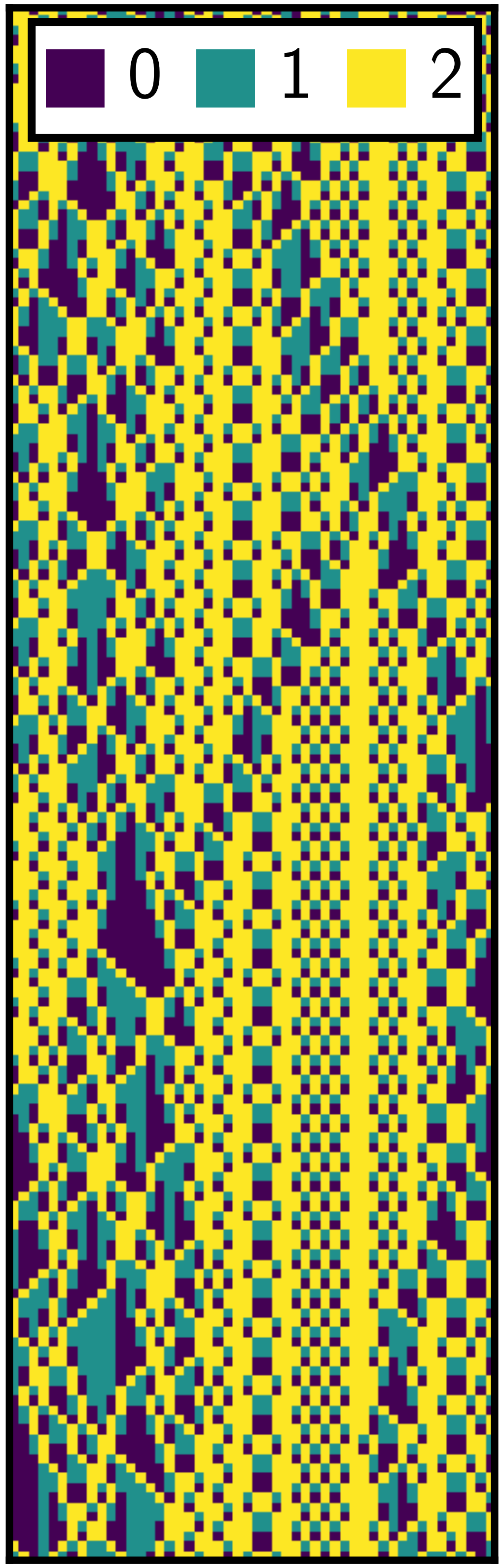

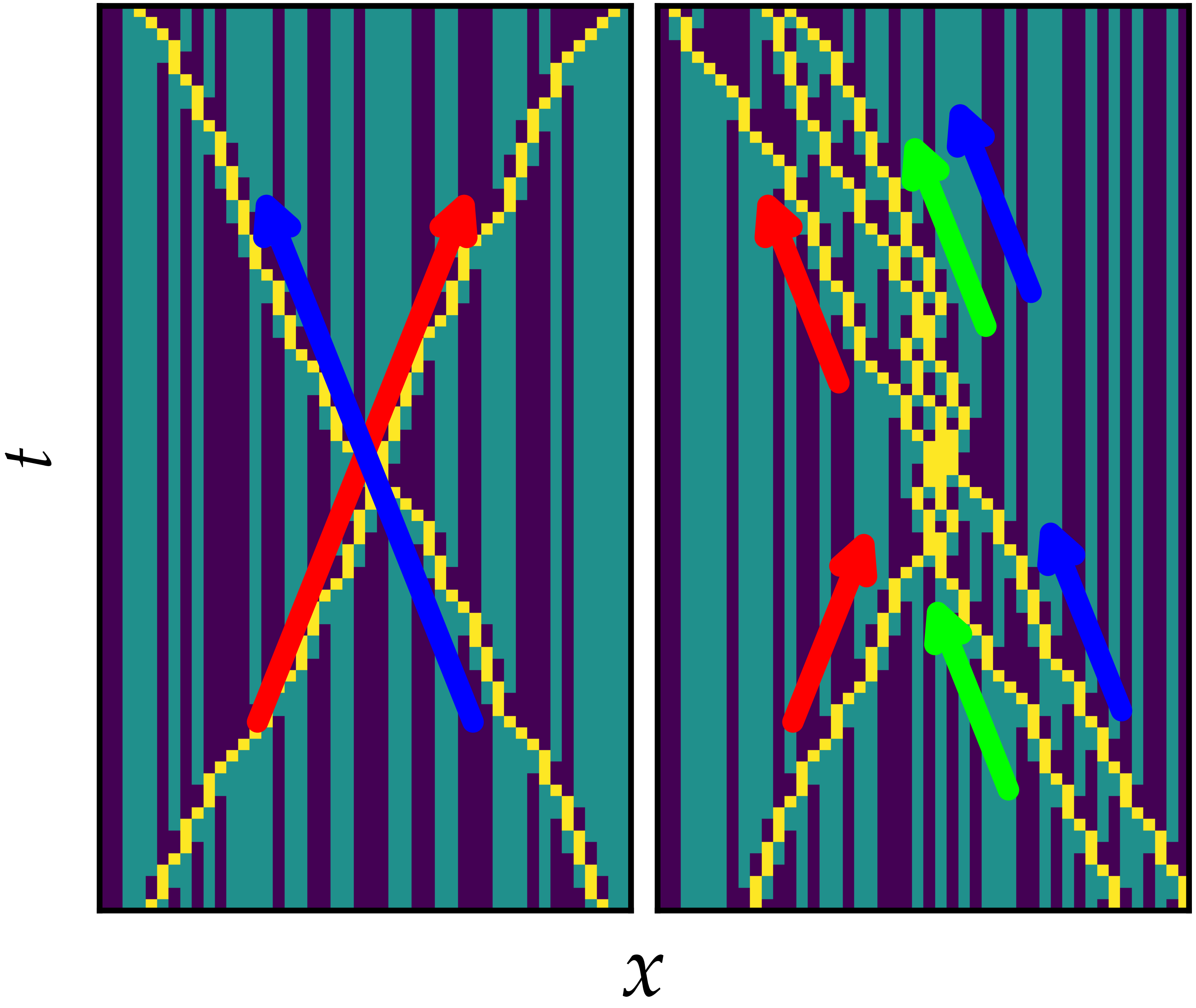

Fermi–Pasta–Ulam–Tsingou behaviour

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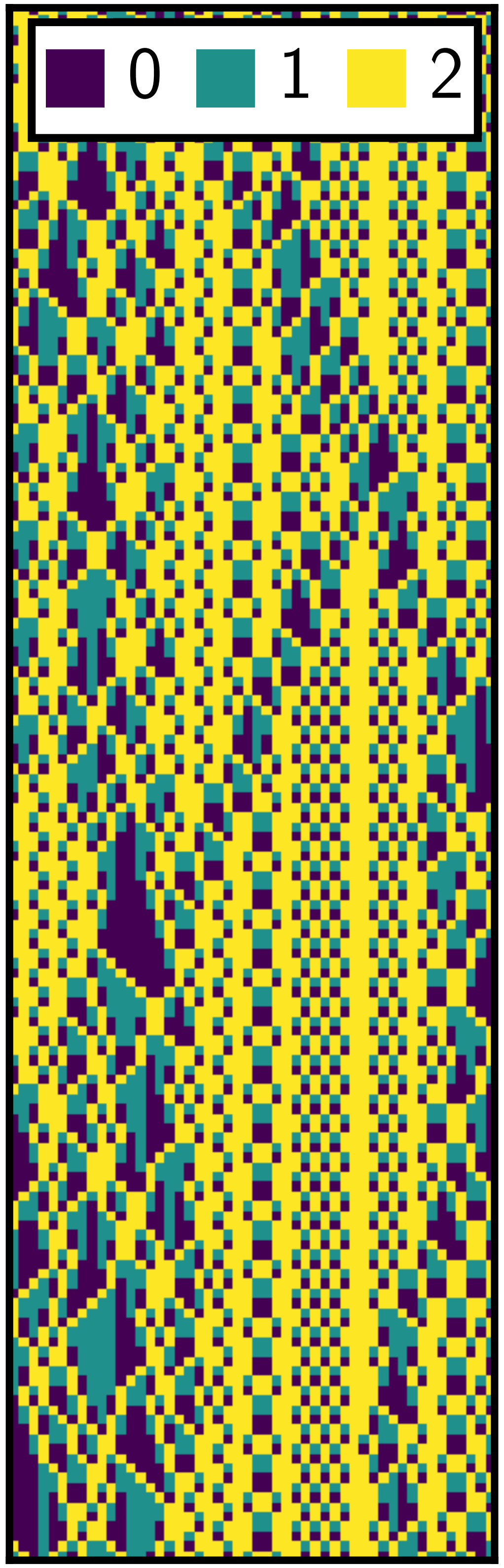

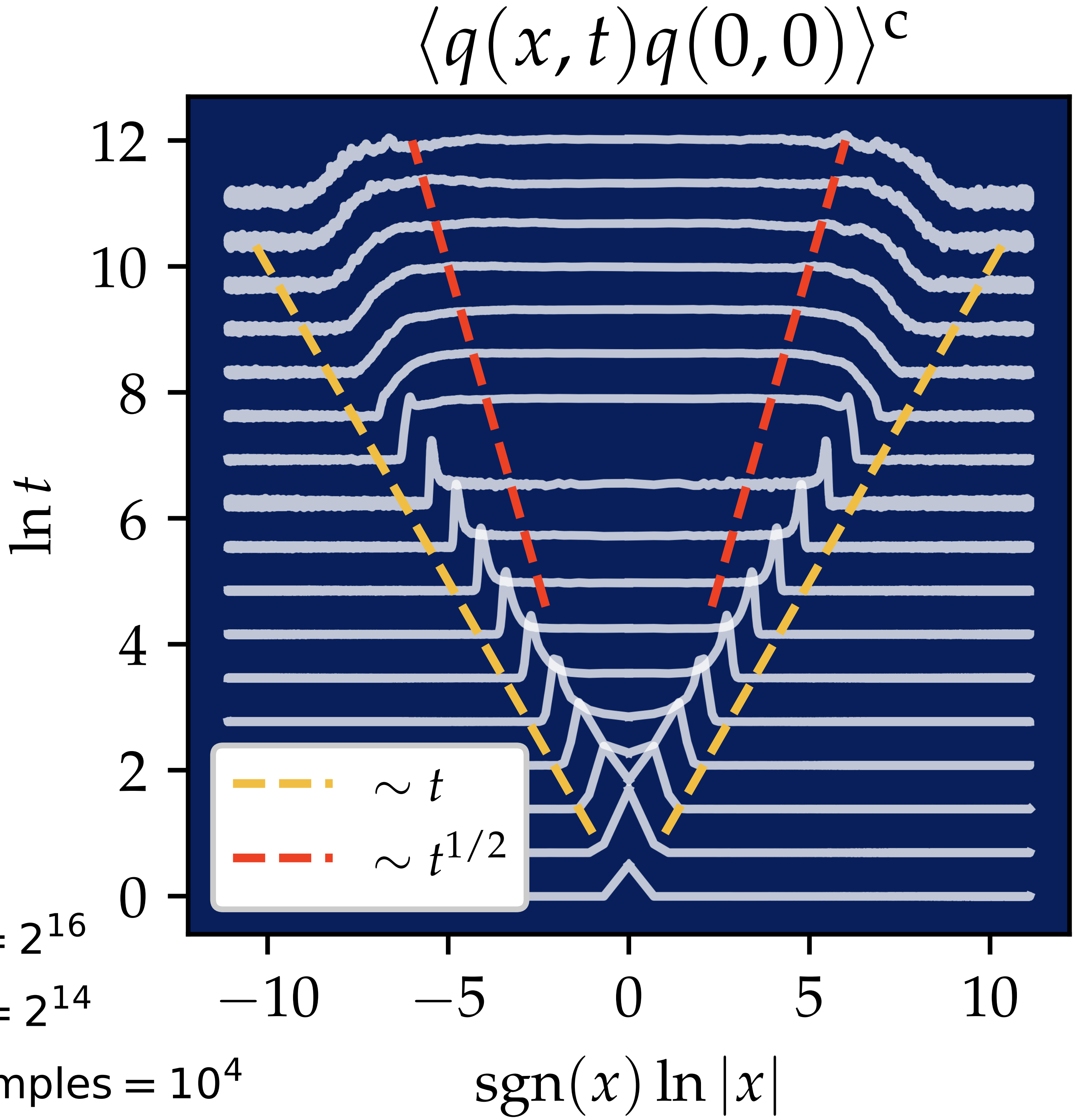

$L=2^{16}$
$T=2^{14}$
$\mathrm{samples}=10^4$

SUMMARY

Chaotic classical deterministic circuits → general platform for HD

Straightforward to generalise to D>1, quantum, stochastic …

Related work:

- [Sharipov-Koterle-Grozdanov-Prosen arXiv:2503.16593]
- Do not do HD, but classifies most d=3 gates, finds symmetries, recurrence times, CQs (including quasi-local), also sees KPZ

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arXiv:2503.08788

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MLiS Machine Learning in Science School of Physics & Astronomy and Faculty of Science